Geometric Proof

Geometry begins with **Undefined Terms: Point, Line, and Plane.**

Using these three terms, all other geometric terms can be defined.

Practice: Try to define these three terms using point, line, and plane.
   Ray
   Circle
   Angle
   Polygon

Definitions require no proof, they are simply descriptions of geometric terms.

   Ray: The set of all points on a line beginning at an endpoint and extending forever in one direction.
   Circle: The set of all points on a plane that are a given distance (radius) from a given point (center).
   Angle: Formed by two rays which share a common endpoint.
   Polygon: A plane figure enclosed by at least three line <u>segments</u> which intersect only at their endpoints.

Definitions build on one-another to describe more complex geometric figures.

Properties are often learned in Algebra.
They are the basic rules of arithmetic and equality. For example, the commutative property of addition states that order does not matter for addition.
a + b = b + a

Aritmetic and equality properties you will need for proofs:

Arithmetic Properties:
Commutative, Associative, and Distributive properties (including FOIL).

Equality Properties:
Reflexive, Transitive (aka Substitution), Symmetric, Addition, Subtraction, Multiplication, Division properties of equality.
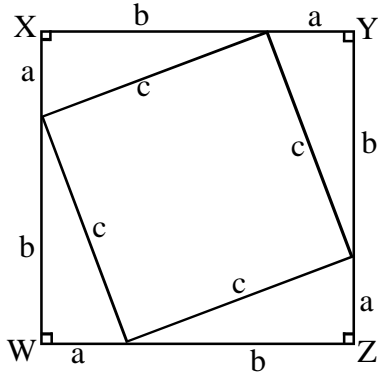
Each of the properties of equality for algebra can be used as a congru-ence property for geometry.

Geometric Proof

Using the arithmetic properties:

The diagram below can be used to prove the Pythagorean Theorem.
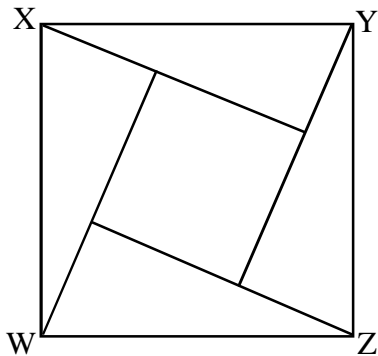Write the short algebraic Proof in two columns, showing $a^2+b^2=c^2$.

Given:
Area of square WXYZ $= (a+b)^2$
and
Area of square WXYZ $= 2ab+c^2$
(this is the sum of the inner square
and 4 triangles)

Practice:
This diagram works as well.
Use it to write an algebraic proof of
the Pythagorean Theorem.

Terms, Definitions, and Properties are stil not enough for most
Proofs. In order to begin a proof, you must have a set of accepted truths
called **Postulates.**

Euclid developed the most famous set of 5 postulates:
1. Two points determine a line.
2. Any line segment can be extended in either direction as far as desired.
3. Given any length and any point, a circle can be drawn with the length
as its radius and the point as its center.
4. All right angles are congruent.
5. Two lines which intersect a third line so that two interior angles o none
side of the line equal less than two right angles, the lines will meet on that
side.

Using only the first four postulates, Euclid was able to prove the first 28
theorems in his book *Elements,* written around 2300 years ago and laying
the foundation for geometry.

Postulates

Postulates are assumed to be true without proof.
Examples:
Line intersection postulate: Two lines intesect at exactly one point.
Angle bisector postulate: Every angle has exactly one bisector.

Theorems are statements that can be proven using terms, definitions, properties, and postulates.
Examples: The Pythagorean Theorem or the Triangle Sum Theorem.

Many different lists of postulates exist, each giving a starting point for proof writing. Any postulate which can be proven using one or more of the other postulates is unnecessary, and is considered a theorem.

Postulates to be used when writing proofs:

Line Postulate: Two points define a line.
Line Intersection Postulate: The intersection of two lines is a point.
Segment Duplication Postulate: You can duplicate segments.
Angle Duplication Postulate: You can duplicate angles.
Midpoint Postulate: Every segment has exactly one midpoint.
Angle Bisector Postulate: Every angle has exactly one bisector.
Parallel Postulate: Exactly one parallel line can be constructed to a line through a point not on that line.
Perpendicular Postulate: Exactly one perpendicular line can be constructed to a line through a point not on that line.
Segment Addition Postulate: If point B is on segment AC between A and C, AB + BC = AC.
Angle Addition Postulate: If point D lies within angle ABD, then $\angle ABD + \angle DBC = \angle ABC$ (adjacent angles add up)
Linear Pair Postulate: If two angles are a linear pair, then they are supplementary.
Corresponding Angles Postulate
SSS, ASA, and SAS Congruence Postulates

Answer:
Why don't we need:
Vertical Angles Postulate?
AIA Angles Postulate?
Triangle Sum Postulate?

Theorems

Theorems are statements that have been proven. Like the building of complex definitions using simpler ones, more complex theorems can be build using previously proven ones.

Below is a list of some basic theorems that we have covered and may be used in your proof writing:

Triangle Sum Theorem (angles = 180°)
SAA Congruence Theorem
Vertical Angles Theorem
AIA Theorem (ASA, SSS, and SAS are postulates in our text.)
Congruent and Supplementary Theorem (congruent + supplementary = right angles)

Many more can be used. In general, you should only use a theorem in a proof if we have already proven it, and obviously you should never use a theorem to prove itself!

**Before you begin a proof, you should:**
1. Create a diagram showing all givens.
2. State what is given and what must be shown.
3. Plan before you begin… find the shortest route from A to B.
4. Formalize a proof.

Example:
Prove that if two parallel lines are cut by a transversal, then the interior angles on the same side of the transversal are supplementary.
(Interior Supplements Theorem)

1. Diagram:
2. Givens:
       Prove:
3. Plan:
       What postulates and
       theorems will be useful?
4. Two-Column Proof:
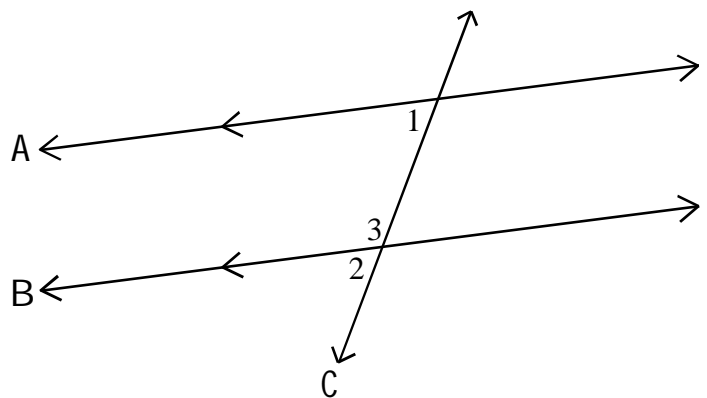
Proofs

Triangle congruence is useful in many proofs.
SAS, ASA, and SSS are congruence postulates.

Begin by proving the AAS postulate using the triangle
sum theorem and the ASA postuate.

1. Diagram:
2. Givens:
Prove:
3. Plan:
What postulates and
theorems will be useful?
4. Two-Column Proof:

Practice:
Prove that the measure of a triangle's exterior angle is congruent to the
sum of its remote interior angles.

1. Diagram:
2. Givens:
Prove:
3. Plan:
What postulates and
theorems will be useful?
4. Two-Column Proof:

Practice:
In an isosceles triangle, the altitudes perpendicular to congruent sides are
congruent.

1. Diagram:
2. Givens:
Prove:
3. Plan:
What postulates and
theorems will be useful?
4. Two-Column Proof:

Quadrilateral Proofs

Remember your quadrilateral definitions: All have four sides.

Square: Opposite sides parallel. All sides congruent.
   All angles congruent.
Rectangle: Opposite sides parallel. All angles congruent.
Rhombus: Opposite sides parallel. All sides congruent.
Parallelogram: Opposite sides parallel.
Trapezoid: Exactly one pair of parallel sides.
Kite: Two pairs of consecutive congruent sides.

Do not assume additional information for the definition of each.
"Opposite sides parallel." is redundant for rectangles and squares. Why?
Also, it is easy enough to prove that the opposite sides of a parallelogram
are congruent: (Prove it, this applies to all parallelograms.)

We can use the **Opposite Sides Theorem** for all parallelograms from
now on.

More Theorems that we can use:

Opposite Angles Theorem (opposite angles of parallelogram are cong.)
Converse of AIA (lines are parallel if AIA are congruent)
Converse of Interior Supplements Theorem (lines are parallel)
Converse of Corresponding Angles Theorem (lines are parallel)

Practice Proofs:

1. Prove that the sum of the angles in a quadrilateral is $360°$.

2. Prove that the diagonals of an isosceles trapezoid are congruent.

3. Prove that the nonvertex angles in a kite are congruent.

4. Prove that a quadrilateral with opposite sides congruent is a
   parallelogram.

Converse Inverse Contrapositive

If/Then Statements in Geometry (and mathemetics in general) are called **Conditionals.**

You can write most postulates and theorems in geometry as conditional statements. Try to write the Pythagoren Theorem as a conditional in If/Then form.

Try to write the following theorem/postulates as conditionals:
AIA Theorem
SSS Congruence Postulate
Linear Pair Postulate

You can write conditionals that are not math related:
If an object is less dense than water, then the object will float.
If it is raining, then Jerry always has his umbrella.

Try to convert each statement into a conditional.
1. The sum of two odd numbers is always even.
2. Hens lay eggs.

Conditionals can be expressed with symbols.
If you add two odd integers (A) then the sum will be even (B).
A implies B: A$\rightarrow$B

The **converse** of a statement switches A and B:
If the sum of two integers is even then you added two odd integers.
B implies A: B$\rightarrow$A (the converse of a true statement is not always true)

The **inverse** of a statement negates A and B:
If you add two integers that are NOT ODD then the sum will NOT be even.
NOT A implies NOT B: ~B$\rightarrow$~A (~ means NOT)

The **contrapositive** of a conditional switches and negates:
If the sum of two integers is NOT even then you have NOT added two odd integers.
NOT B implies NOT A: ~B$\rightarrow$~A
The contrapositive of a mathematically true statement is TRUE. This is important in proofs because if you can prove the contrapositive, you can prove the original statement. Write the converse, inverse, and contrapositive of each theorem/postulate below. Label each as true/false/invalid.
1. AIA Theorem   2. SSS Congruence Postulate   3.Linear Pair Postulate

Conditionals Practice

Geometry 13+

For each:
1. Convert statement into a conditional (if/then) statement.
2. Write the converse, inverse, and contrapositive for each.
3. Label each logical conclusion as true, false, or invalid (not necessarily T/F).
   (Assume that each statement is true, especially number 4.)

1.
Mammals have fur.

Statement: _____ True

Converse: _____ T/F/Inv _____

Inverse: _____ T/F/Inv _____

Contrapositive: _____ T/F/Inv _____

2.
When the volume of a gas decreases, its pressure increases. (All other variabes are unchanged.)

Statement: _____ True

Converse: _____ T/F/Inv _____

Inverse: _____ T/F/Inv _____

Contrapositive: _____ T/F/Inv _____

3.
A fire will only burn if there is oxygen present.

Statement: _____ True

Converse: _____ T/F/Inv _____

Inverse: _____ T/F/Inv _____

Contrapositive: _____ T/F/Inv _____

4.
Students who don't complete their conditionals practice are losers.

Statement: _____ True

Converse: _____ T/F/Inv _____

Inverse: _____ T/F/Inv _____

Contrapositive: _____ T/F/Inv _____

Coordinate Proof   Geometry 13+

Coordinate Proofs rely primarily on slope, midpoint, and distance formulas for proof.

Recall the following:

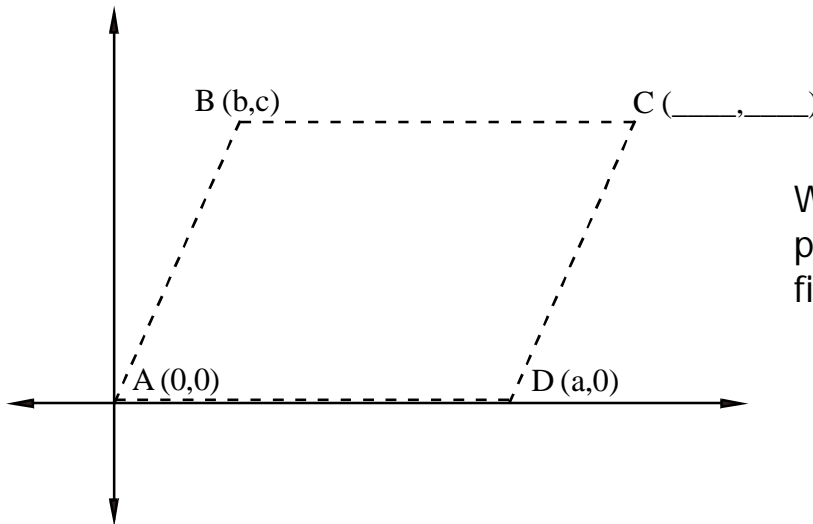**Distance Formula:**   **Midpoint Coordinates:**

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$ $$= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

The most important thing about setting up a coordinate proof is labeling the coordinates correctly.

Ex. Prove that the diagonals of a parallelogram bisect each other.

What coordinates for point C would make this figure a parallelogram?

It is important that the diagram is accurate: For ABCD to be a parallelogram, opposite sides must be parallel. How would you check this?

Proving that the diagonals bisect each other is as easy as showing that the midpoint of segment AC is the same as the midpoint of segment BD.

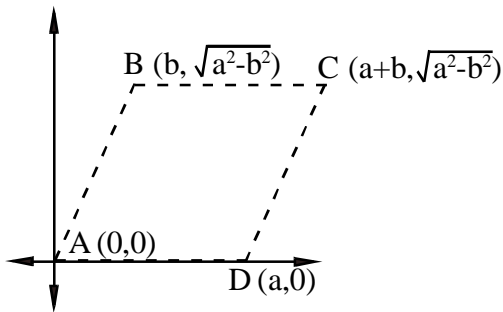**Practice:** Fill-in the missing coordinates below to create a rhombus:

Coordinate Proof

Practice: Prove that each of the following is true using coordinate proof:

1. The diagonals of a rhombus are perpendicular.
(Prove that the slopes of the diagonals are negative reciprocals: requires clever use of Algebra.)

B $(b, \sqrt{a^2-b^2})$ C $(a+b, \sqrt{a^2-b^2})$

A $(0,0)$

D $(a,0)$

2. Connecting the midpoints of a rhombus will always form a rectangle. (Add some midpoints, find the slope of the midsegments. Zero slope = horizontal, undefined slope = vertical).

B $(0,b)$

C $(a,0)$

A $(-a,0)$

D $(0,-b)$
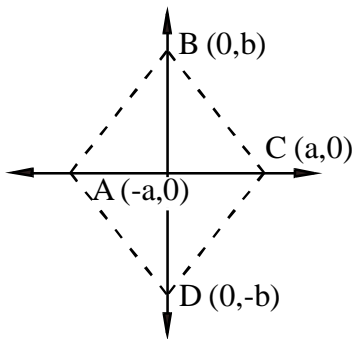
Some Proof Writing Terms To Use

Properties:
Commutative, Associative, and Distributive properties (including FOIL).
Reflexive, Transitive/Substitution, Symmetric, Addition, Subtraction, Multiplication, Division (properties of equality).
Multiplicative Inverse, Multiplicative Identity, Additive Inverse, Additive Identity.

Postulates:
Line Postulate
Line Intersection Postulate
Segment Duplication Postulate
Angle Duplication Postulate
Midpoint Postulate
Angle Bisector Postulate
Parallel Postulate
Perpendicular Postulate
Segment Addition Postulate
Angle Addition Postulate
Linear Pair Postulate
Corresponding Angles Postulate
SSS, ASA, and SAS Congruence Postulates

Theorems:
Vertical Angles Theorem
AIA Theorem
Triangle Sum Theorem
Isosceles Base Angle Theorem
Pythagorean Theorem
Parallelogram Sides Theorem
Kite Diagonals Theorem
Parallelogram Diagonals Theorem
Exterior Angle Theorem (for triangles)

Definition of:
Isosceles, Equilateral
Parallel, Perpendicular
Parallelogram, Kite, Rhombus, Square, Rectangle
Congruent

Practice Quiz

Geometry 13

Fill-in the blanks to complete the following proof.
Given:
Isosceles Triangle ABC with AB = BC, and angle bisectors AX and CY (with X on BC and Y on AB).

Prove:
BX = BY

|	
△ ABC is isosceles	1. _____
∠BAC = ∠BCA	2. _____
½∠BAC = ½∠BCA	3. _____
$\overline{AX}$ bisects ∠BAC	4. _____
$\overline{CY}$ bisects ∠BCA	5. _____
∠BCY = ½∠BCA	6. _____
∠BAX = ½∠BAC	7. _____
∠BCY = ∠BAX	8. _____
$\overline{AB}$ = $\overline{BC}$	9. _____
∠B =∠B	10. _____
△BAX = △BCY	11. _____
BX = BY	12. _____

Label each as the converse, inverse, and contrapositive of the following conditional: If a radius bisects a chord, then the radius and chord are perpendicular.

13. If a radius and chord are not perpendicular, then
the radius does not bisect the chord.   13. _____

15. If a radius does not bisect a chord, then the radius
and chord are not perpendicular.   14. _____

14. If a radius and a chord are perpendicular, then
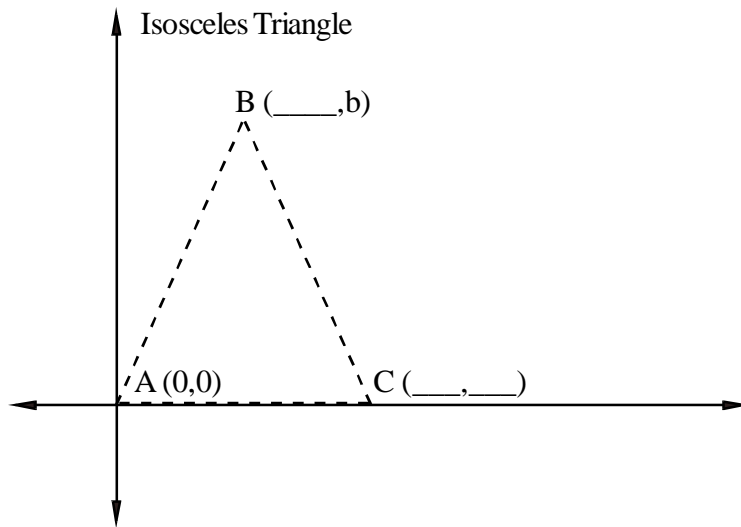the radius bisects the chord.   15. _____

Name_____ Period _____

Practice Quiz

Geometry 13

Complete the following diagrams with the missing coordinate pairs:

B (___,___)

Rhombus

A (0, __)

C (a,0)

D (0,-b)
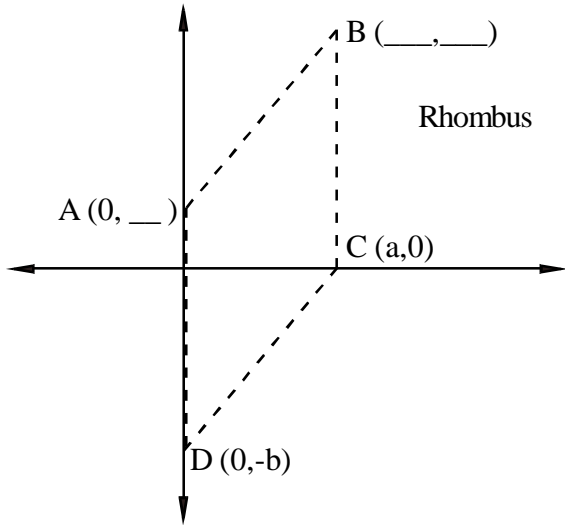
16. A. _____

17. B. (x-coord.) _____

18. B. (y-coord.) _____

Isosceles Triangle

B (_____,b)

A (0,0)   C (___,___)

19. B. _____

20. C. _____

Complete the following diagrams with the missing points:

Square

B (a,___)

C (___,___)

A (0,0)

D (b,___)
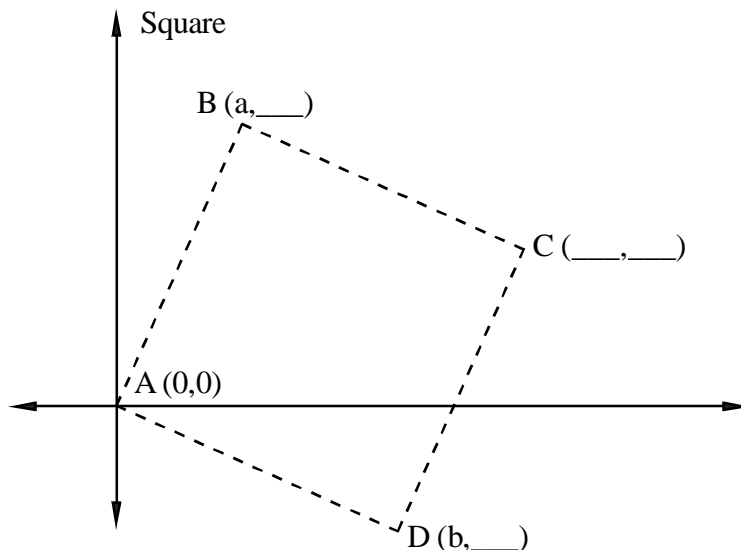
21. B. _____

22. C. _____

23. D. _____

24. midpoint of BD

_____