2.7: Combinations

Consider the following examples:

Example: There are six scrabble letters left in the bag at the end of the game (FHJSU and Y). If you reach in and grab two letters, how many different pairs of letters are possible?

Reasoning: There are six possible letters to choose from. The first letter you pick can be any one of six letters, and the second can be any one of the remaining five. 6 times 5 appears to give us 30 choices, however, the order in which we choose the letters is not important (FH is the same as HF). We divide by the number of ways we can arrange the chosen letters (2! = 2) to get 30/2 = 15 possible pairs of letters.

Example: Remy wants to offer 3 sodas at his snack stand. He has a list of 8 sodas to choose from. How many combinations of 3 sodas are possible?

Reasoning: There are $8 \cdot 7 \cdot 6$ possible ways to select three sodas in order, and $3 \cdot 2 \cdot 1$ possible ways to order the three selected sodas (selecting Coke, Sprite, and Root Beer is the same as Sprite, Root Beer, and Coke). Dividing out the duplicate outcomes gives us:

$$\frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = 56 \text{ combinations of sodas.}$$

Example: Roger has won a contest at the fair, and gets to choose four *different* prizes from a set of nine. How many combinations of four prizes can he choose from a set of nine?

Reasoning: There are $9 \cdot 8 \cdot 7 \cdot 6$ possible ways to select four prizes in order, and $4 \cdot 3 \cdot 2 \cdot 1$ possible orders. Dividing, we get:

$$\frac{9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 3 \cdot 2 \cdot 1} = 126 \text{ prize combinations.}$$

Combinations:

The primary difference between **combinations** and permutations is that with combinations, *order does not matter*. There is a formula for combinations and some notation which we must consider. C(n,r) asks for the number of ways n things can be taken r at a time.

For example, choosing four prizes from a set of nine is expressed:

$$C(9,4) = \frac{9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 3 \cdot 2 \cdot 1} = 126 .$$

The formula for $C(n,r)$: $C(n,r) = \dfrac{n!}{r!(n-r)!}$.

Notation: $C(n,r)$ is also notated $\binom{n}{r}$ or nCr and is often called the "choose" function, for example $_9C_4$ is read "nine choose four". I will use all three notations interchangeably so that you can become familiar with each.

Check for Understanding:

2.011 Find $_6C_2$.

2.111 Find $C(6,4)$.

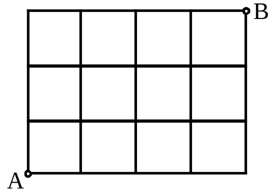
2.211 Find the value of $C(85,25)$ subtracted from $C(85,60)$.

2.311 How many different sets of three books can be chosen from a shelf of 20?

2.411 Eight people are asked to select a leadership team of three members. How many different leadership teams are possible?

Paths on a Grid:
Tracing routes across grids is a great way to show how combinations can arise in less obvious problems.

Example: Tracing the lines starting from point A on the unit grid below, how many distinct 7-unit paths are there from A to B?

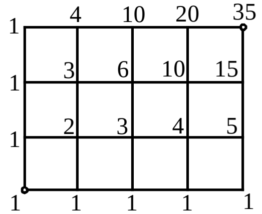


Reasoning (1): Take any path and represent it as series of moves, either U (up) or R (right). Every unique 7-unit path consists of four R's and three U's. The question then becomes: How many ways can we rearrange the letters RRRRUUU:

$$\frac{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{(3\cdot 2\cdot 1)(4\cdot 3\cdot 2\cdot 1)} = 35.$$

Reasoning (2): We must move up 3 times in 7 moves. ${}_7C_3 = 35$ possible combinations of up moves. Think of this as choosing where to place the three U's in 7 blanks, then filling the remaining blanks with R's.

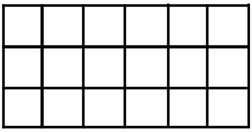
Reasoning (3): We can also show this by counting the number of ways there are to get to each point.



While not recommended for this problem, this technique is useful for more complex patterns where the grid is irregular or incomplete.

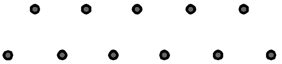
Combinations: Beyond Casework.
Combinations and the "choose" function offer shortcuts for many other seemingly complex problems.

Example: How many distinct rectangles can be formed by tracing the lines on the grid below?



Reasoning: This would be a casework nightmare, so we look for a better way. Each rectangle requires that we choose two horizontal and two vertical lines. There are ${}_4C_2 = 6$ ways to choose two horizontal lines and ${}_7C_2 = 21$ ways to choose two vertical lines, which gives us $21 \cdot 6 = 126$ rectangles. Good thing we didn't need to count them using casework techniques!

Example: How many distinct triangles can be formed by connecting three of the points below?

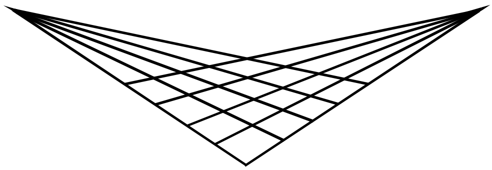


Reasoning: It is tempting to say that there are 11 points and we can choose any three, so ${}_{11}C_3 = 165$ possible triangles. Unfortunately, if we choose three points that are on the same row, they will not form a triangle. There are ${}_5C_3 = 10$ ways to select three points on the top row, and ${}_6C_3 = 20$ ways to select three points on the bottom row. When we subtract these from the total, we find that there are 135 triangles which can be formed.

If a problem looks too difficult for casework, there is likely a counting shortcut that can be applied.

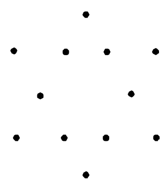
Practice: Beyond Casework

2.511 How many distinct triangles can be traced along the lines in the diagram below?

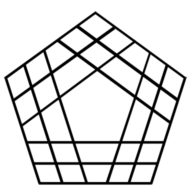


2.611 John walks six blocks on a city grid of sidewalks to his favorite deli for lunch: three blocks north and three blocks west. He never uses the exact same path on his return to work. If John always stays on city sidewalks and goes six blocks each way, how many different ways can John walk to the deli and back?

2.711 Twelve points are arranged into six rows of four points each in the shape of a six-pointed star. How many ways can three of these points be connected to form a triangle?



2.811 How many pentagons can be formed by tracing the lines of the figure below?



2.911 A regular dodecahedron has 12 pentagonal faces, 20 vertices, and 30 edges. How many of the triangles which can be formed by connecting three of its vertices have at least one side within the dodecahedron?