4.651 900/n is a positive integer only when n is a factor of 900. $900 = 2^2 \cdot 3^2 \cdot 5^2$ and has 27 factors. There are **27** values of n which make 900/n a positive integer.

4.652 The tires have circumference 27π inches and 30π inches. The LCM (270π) inches gives us the distance after which the tires will both reach their original position in the rotation. The stars will be positioned as shown in the drawing 5 times (every 54π inches) during this period. 1/4 mile = 15,840 inches. $15{,}840/(54\pi) = 93.4$. If the wheels begin in the position shown, this makes **94** times.

4.671 $9^6 = (3^2)^6 = 3^{12} = \mathbf{1{,}000{,}000{,}000{,}000_3}$.

4.681 When we borrow in base 10 we "borrow" 10, but in base 6 we "borrow" 6.

$$\begin{array}{r} {\scriptstyle 1\ 6\ 9} \\ 213_6 \\ -\ 144_6 \\ \hline \mathbf{25_6} \end{array}$$

4.690 The two values are both equal to $(1 + 2)(1 + 3)(1 + 5)(1 + 7)(1 + 11)$. Their quotient is **1**.

4.720 I like to start these problems with the largest number possible (9,876,543,210) and modify it as necessary. Looking at the number, it is already divisible by 9 and 10, but not by 8 and 11. For 11, the digit sum is 45, so the sum of alternating digits cannot be equal. The difference must be 11 with alternating digit sums of 17 and 28. 9,876,543,210 has alternating digit sums of 25 and 20, so we must increase the digit sum of 9 + 7 + 5 + 3 + 1 to 28. Trading the 1 for a 4 is the best way to do this (9 + 7 + 5 + 3 + 4 = 28). Leave the largest five digits in order on the left and the zero on the right. Look for arrangements of the underlined digits which make the number divisible by 8 while maintaining the alternating digit sums. 9,876,524,130 is not divisible by 8, neither is 9,876,523,140, but **9,876,513,240** is.

4.722 We have a single misplaced digit which we will call x and a 5-digit number y. In the correct order, we have $10y + x$, but in the incorrect order written by Michael we have $100{,}000x + y$. The incorrect integer is 5 times the correct integer, giving us $5(10y+x)=100{,}000x+y$ which simplifies to $49y=99{,}995x$. Dividing by 7 we get $7y=14{,}285x$. We can use $x = 7$ and $y = 14{,}285$ which makes Michael's number **714,285**. Alternatively (as in problem **4.622**) the problem suggests a cyclic number found in the repeating block of digits in sevenths:

$$\frac{1}{7}=0.\overline{142857} \text{ and } \frac{5}{7}=0.\overline{714285}.$$

4.731 For this problem I will introduce a function called the floor value. The floor value of a number, indicated by $\lfloor x \rfloor$ represents the greatest integer value of x and basically means round down. For example: $\lfloor 9.8 \rfloor = 9$ and $\lfloor 73/11 \rfloor = 6$. To find the power of 7 in 2,500!

we add $\left\lfloor \frac{2{,}500}{7} \right\rfloor + \left\lfloor \frac{2{,}500}{7^2} \right\rfloor + \left\lfloor \frac{2{,}500}{7^3} \right\rfloor + \left\lfloor \frac{2{,}500}{7^4} \right\rfloor$.

$\left\lfloor \frac{2{,}500}{7} \right\rfloor = 357$, which means there are 357 multiples of 7 in 2,500! Each contributes a 7 to the prime factorization of 2,500! Continuing, there are 51 multiples of 7^2, and each contributes an additional 7. There are 7 multiples of 7^3 and 1 multiple of 7^4. This gives us a total of 357 + 51 + 7 + 1 = 416 sevens in the prime factorization of 2,500!, making 416 ($\mathbf{7^{416}}$) the power of 7 in 2,500!.

4.742 The units digits in the powers of 2 repeat: 2-4-8-6-2-4-8-6..., and 222/4 leaves a remainder of 2. This means that 222^{222} must end in a 4. Multiplying this result by 9 would give us a units digit **6**.

4.750 For a multiple of 6 to have 9 factors, it must be a perfect square, making our integer $2^2 \cdot 3^2$. Multiplying by 10 we get $2^3 \cdot 3^2 \cdot 5$, which has **24** factors.

4.751 We use the floor function again as explained in **4.731**:

$$\left\lfloor \frac{343}{7} \right\rfloor + \left\lfloor \frac{343}{7^2} \right\rfloor + \left\lfloor \frac{343}{7^3} \right\rfloor = 49 + 7 + 1 = \mathbf{57,\ or\ 7^{57}}.$$

4.752 For any pair of consecutive integers, one must be even and therefore divisible by 2, so we look for the least positive integer whose prime factorization is of the form $2^2 \cdot a$ or $a^2 \cdot 2$ for which an adjacent integer has 6 factors. $2^2 \cdot 11 =$ **44** and $3^2 \cdot 5 = 45$ are the first pair of consecutive integers which have exactly 6 factors each. Pairs like (75,76) and (98,99) work as well.

4.771 0.24_5 is $2(5^{-1}) + 4(5^{-2}) = \frac{2}{5} + \frac{4}{25} = \frac{14}{25} = \mathbf{0.56}$.

4.781 Begin with the subtraction shown on the right. To multiply by 100_6, just add two zeros to get $\mathbf{2{,}400_6}$.

$$\begin{array}{r} 6\;9\;\, \\ \not{1}\not{1}\not{3}_6 \\ -\ 45_6 \\ \hline 24_6 \end{array}$$

4.790 $496 = 16 \cdot 31 = 2^4 \cdot 31$ which has a factor sum $(1 + 2 + 4 + 8 + 16)(1 + 31) = (31)(32) =$ **992**. 496 is a perfect number (**p.147**), a number p whose factor sum is $2p$. There are relatively few known perfect numbers, and all known perfect numbers are of the form: $p = 2^{n-1}(2^n - 1)$ where $2^n - 1$ is prime (called a Mersenne prime). In order for $2^n - 1$ to be prime, n must be prime. The largest known primes are all Mersenne primes.

4.820 456,564,465,645 has a digit sum of 4(4+5+6) = 60, so it is divisible by 3. It is not even, so it is not divisible by 6, but three less than 456,564,465,645 is divisible by 6, so dividing by 6 leaves a remainder of **3**.