THE MATHEMATICAL ASSOCIATION OF AMERICA
American Mathematics Competitions

10th Annual American Mathematics Contest 10

AMC 10
CONTEST A

Solutions Pamphlet

Tuesday, FEBRUARY 10, 2009

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, e-mail, World Wide Web or media of any type during this period is a violation of the competition rules.*

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1. **Answer (E):** Because $\frac{128}{12} = 10\frac{2}{3}$, there must be 11 cans.

2. **Answer (A):** The value of any combination of four coins that includes pennies cannot be a multiple of 5 cents, and the value of any combination of four coins that does not include pennies must exceed 15 cents. Therefore the total value cannot be 15 cents. The other four amounts can be made with, respectively, one dime and three nickels; three dimes and one nickel; one quarter, one dime and two nickels; and one quarter and three dimes.

3. **Answer (C):** Simplifying the expression,

$$1 + \frac{1}{1 + \frac{1}{1+1}} = 1 + \frac{1}{1 + \frac{1}{2}} = 1 + \frac{1}{\frac{3}{2}} = 1 + \frac{2}{3} = \frac{5}{3}.$$

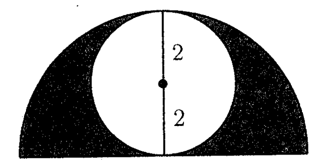
4. **Answer (A):** Eric can complete the swim in $\frac{1/4}{2} = \frac{1}{8}$ of an hour. He can complete the run in $\frac{3}{6} = \frac{1}{2}$ of an hour. This leaves $2 - \frac{1}{8} - \frac{1}{2} = \frac{11}{8}$ hours to complete the bicycle ride. His average speed for the ride must be $\frac{15}{11/8} = \frac{120}{11}$ miles per hour.

5. **Answer (E):** The square of 111,111,111 is

$$\begin{array}{r}
111111111\\
\times\, 111111111\\
\hline
111111111\\
1111111110\\
11111111100\\
111111111000\\
1111111110000\\
11111111100000\\
111111111000000\\
1111111110000000\\
11111111100000000\\
\hline
12345678987654321
\end{array}$$

Hence the sum of the digits of the square of 111,111,111 is 81.

6. **Answer (A):** The semicircle has radius 4 and total area $\frac{1}{2} \cdot \pi \cdot 4^2 = 8\pi$. The area of the circle is $\pi \cdot 2^2 = 4\pi$. The fraction of the area that is not shaded is $\frac{4\pi}{8\pi} = \frac{1}{2}$, and hence the fraction of the area that is shaded is also $\frac{1}{2}$.



7. **Answer (C):** Suppose whole milk is $x\%$ fat. Then 60% of x is equal to 2. Thus

$$x = \frac{2}{0.6} = \frac{20}{6} = \frac{10}{3}.$$

8. **Answer (B):** Grandfather Wen's ticket costs \$6, which is $\frac{3}{4}$ of the full price, so each ticket at full price costs $\frac{4}{3} \cdot 6 = 8$ dollars, and each child's ticket costs $\frac{1}{2} \cdot 8 = 4$ dollars. The cost of all the tickets is $2(\$6 + \$8 + \$4) = \$36$.

9. **Answer (B):** Let the ratio be r. Then $ar^2 = 2009 = 41 \cdot 7^2$. Because r must be an integer greater than 1, the only possible value of r is 7, and $a = 41$.

10. **Answer (B):** By the Pythagorean Theorem, $AB^2 = BD^2 + 9$, $BC^2 = BD^2 + 16$, and $AB^2 + BC^2 = 49$. Adding the first two equations and substituting gives $2 \cdot BD^2 + 25 = 49$. Then $BD = 2\sqrt{3}$, and the area of $\triangle ABC$ is $\frac{1}{2} \cdot 7 \cdot 2\sqrt{3} = 7\sqrt{3}$.

OR

Because $\triangle ADB$ and $\triangle BDC$ are similar, $\frac{BD}{3} = \frac{4}{BD}$, from which $BD = 2\sqrt{3}$. Therefore the area of $\triangle ABC$ is $\frac{1}{2} \cdot 7 \cdot 2\sqrt{3} = 7\sqrt{3}$.

11. **Answer (D):** Let x be the side length of the cube. Then the volume of the cube was x^3, and the volume of the new solid is $x(x+1)(x-1) = x^3 - x$. Therefore $x^3 - x = x^3 - 5$, from which $x = 5$, and the volume of the cube was $5^3 = 125$.

12. **Answer (C):** Let x be the length of $\overline{BD}$. By the triangle inequality on $\triangle BCD$, $5 + x > 17$, so $x > 12$. By the triangle inequality on $\triangle ABD$, $5 + 9 > x$, so $x < 14$. Since x must be an integer, $x = 13$.

13. **Answer (E):** Note that

$$12^{mn} = (2^2 \cdot 3)^{mn} = 2^{2mn} \cdot 3^{mn} = (2^m)^{2n} \cdot (3^n)^m = P^{2n}Q^m.$$

Remark: The pair of integers $(2, 1)$ shows that the other choices are not possible.

14. **Answer (A):** Let the lengths of the shorter and longer side of each rectangle be x and y, respectively. The outer and inner squares have side lengths $y + x$ and $y - x$, respectively, and the ratio of their side lengths is $\sqrt{4} = 2$. Therefore $y + x = 2(y - x)$, from which $y = 3x$.

15. **Answer (E):** The outside square for F_n has 4 more diamonds on its boundary than the outside square for F_{n-1}. Because the outside square of F_2 has 4 diamonds, the outside square of F_n has $4(n-2)+4 = 4(n-1)$ diamonds. Hence the number of diamonds in figure F_n is the number of diamonds in F_{n-1} plus $4(n-1)$, or

$$\begin{aligned} &1 + 4 + 8 + 12 + \cdots + 4(n-2) + 4(n-1) \\ =&1 + 4(1 + 2 + 3 + \cdots + (n-2) + (n-1)) \\ =&1 + 4\frac{(n-1)n}{2} \\ =&1 + 2(n-1)n. \end{aligned}$$

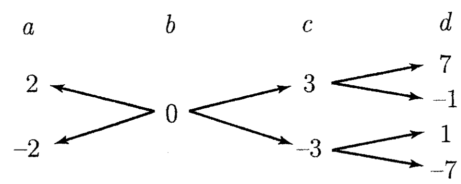
Therefore figure F_{20} has $1 + 2 \cdot 19 \cdot 20 = 761$ diamonds.

16. **Answer (D):** The given conditions imply that $b = a \pm 2$, $c = b \pm 3 = a \pm 2 \pm 3$, and $d = c \pm 4 = a \pm 2 \pm 3 \pm 4$, where the signs can be combined in all possible ways. Therefore the possible values of $|a - d|$ are $2 + 3 + 4 = 9$, $2 + 3 - 4 = 1$, $2 - 3 + 4 = 3$, and $-2 + 3 + 4 = 5$. The sum of all possible values of $|a - d|$ is $9 + 1 + 3 + 5 = 18$.

OR

The equations in the problem statement are true for numbers a, b, c, d if and only if they are true for $a+r$, $b+r$, $c+r$, $d+r$, where r is any real number. The value of $|a - d|$ is also unchanged with this substitution. Therefore there is no

loss of generality in letting $b = 0$, and we can then write down the possibilities for the other variables:



The different possible values for $|a - d|$ are

$$|2 - 7| = 5, \quad |2 - (-1)| = 3, \quad |2 - 1| = 1, \quad |2 - (-7)| = 9.$$

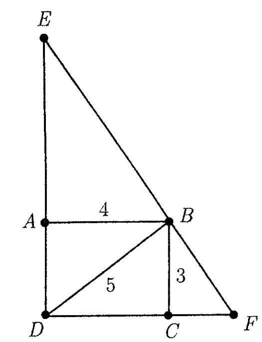
The sum of these possible values is 18.

17. **Answer (C):** Note that $DB = 5$ and $\triangle EBA$, $\triangle DBC$, and $\triangle BFC$ are all similar.

Therefore $\frac{4}{EB} = \frac{3}{5}$, so $EB = \frac{20}{3}$. Similarly, $\frac{3}{BF} = \frac{4}{5}$, so $BF = \frac{15}{4}$.

Thus

$$EF = EB + BF = \frac{20}{3} + \frac{15}{4} = \frac{125}{12}.$$



18. **Answer (D):** For every 100 children, 60 are soccer players and 40 are non-soccer players. Of the 60 soccer players, 40% or $60 \times \frac{40}{100} = 24$ are also swimmers, so 36 are non-swimmers. Of the 100 children, 30 are swimmers and 70 are non-swimmers. The fraction of non-swimmers who play soccer is $\frac{36}{70} \approx .51$, or 51%.

19. **Answer (B):** Circles A and B have circumferences 200π and $2\pi r$, respectively. After circle B begins to roll, its initial point of tangency with circle A touches circle A again a total of

$$\frac{200\pi}{2\pi r} = \frac{100}{r}$$

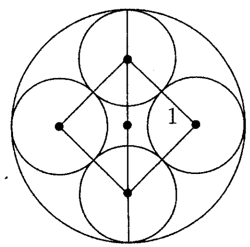
times. In order for this to be an integer greater than 1, r must be one of the integers 1, 2, 4, 5, 10, 20, 25, or 50. Hence there are a total of 8 possible values of r.

20. **Answer (D):** Let r be the rate that Lauren bikes, in kilometers per minute. Then $r + 3r = 1$, so $r = \frac{1}{4}$. In the first 5 minutes, the distance between Andrea and Lauren decreases by $5 \cdot 1 = 5$ kilometers, leaving Lauren to travel the remaining 15 kilometers between them. This requires

$$\frac{15}{\frac{1}{4}} = 60$$

minutes, so the total time since they started biking is $5 + 60 = 65$ minutes.

21. **Answer (C):** It may be assumed that the smaller circles each have radius 1. Their centers form a square with side length 2 and diagonal length $2\sqrt{2}$. Thus the diameter of the large circle is $2 + 2\sqrt{2}$, so its area is $(1+\sqrt{2})^2\pi = (3+2\sqrt{2})\pi$. The desired ratio is



$$\frac{4\pi}{(3+2\sqrt{2})\pi} = 4(3-2\sqrt{2}).$$

22. **Answer (D):** Suppose that the two dice originally had the numbers 1, 2, 3, 4, 5, 6 and 1′, 2′, 3′, 4′, 5′, 6′, respectively. The process of randomly picking the numbers, randomly affixing them to the dice, rolling the dice, and adding the top numbers is equivalent to picking two of the twelve numbers at random and adding them. There are $\binom{12}{2} = 66$ sets of two elements taken from $S = \{1,1',2,2',3,3',4,4',5,5',6,6'\}$. There are 4 ways to use a 1 and 6 to obtain 7, namely, $\{1,6\}$, $\{1,6'\}$, $\{1',6\}$, and $\{1',6'\}$. Similarly there are 4 ways to obtain the sum of 7 using a 2 and 5, and 4 ways using a 3 and 4. Hence there are 12 pairs taken from S whose sum is 7. Therefore the requested probability is $\frac{12}{66} = \frac{2}{11}$.

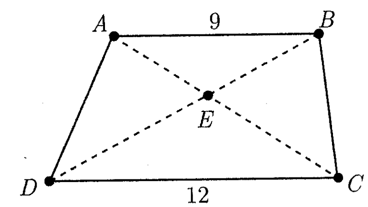
OR

Because the process is equivalent to picking two of the twelve numbers at random and then adding them, suppose we first pick number N. Then the second choice must be number $7-N$. For any value of N, there are two "removable numbers" equal to $7-N$ out of the remaining 11, so the probability of rolling a 7 is $\frac{2}{11}$.

23. **Answer (E):** Because $\triangle AED$ and $\triangle BEC$ have equal areas, so do $\triangle ACD$ and $\triangle BCD$. Side $\overline{CD}$ is common to $\triangle ACD$ and $\triangle BCD$, so the altitudes from A and B to $\overline{CD}$ have the same length. Thus $\overline{AB} \parallel \overline{CD}$, so $\triangle ABE$ is similar to $\triangle CDE$ with similarity ratio

$$\frac{AE}{EC} = \frac{AB}{CD} = \frac{9}{12} = \frac{3}{4}.$$

Let $AE = 3x$ and $EC = 4x$. Then $7x = AE + EC = AC = 14$, so $x = 2$, and $AE = 3x = 6$.



24. **Answer (C):** A plane that intersects at least three vertices of a cube either cuts into the cube or is coplanar with a cube face. Therefore the three randomly chosen vertices result in a plane that does not contain points inside the cube if and only if the three vertices come from the same face of the cube. There are 6 cube faces, so the number of ways to choose three vertices on the same cube face is $6 \cdot \binom{4}{3} = 24$. The total number of ways to choose the distinct vertices without restriction is $\binom{8}{3} = 56$. Hence the probability is $1 - \frac{24}{56} = \frac{4}{7}$.

25. **Answer (B):** Note that $I_k = 2^{k+2} \cdot 5^{k+2} + 2^6$. For $k < 4$, the first term is not divisible by 2^6, so $N(k) < 6$. For $k > 4$, the first term is divisible by 2^7, but the second term is not, so $N(k) < 7$. For $k = 4$, $I_4 = 2^6(5^6+1)$, and because the second factor is even, $N(4) \geq 7$. In fact the second factor is a sum of cubes so

$$(5^6+1) = ((5^2)^3 + 1^3) = (5^2+1)((5^2)^2 - 5^2 + 1).$$

The factor $5^2 + 1 = 26$ is divisible by 2 but not 4, and the second factor is odd, so $5^6 + 1$ contributes one more factor of 2. Hence the maximum value for $N(k)$ is 7.

The problems and solutions in this contest were proposed by Steve Blasberg, Thomas Butts, Steven Davis, Steve Dunbar, Douglas Faires, Jerrold Grossman, John Haverhals, Elgin Johnston, Joe Kennedy, Bonnie Leitch, David Wells, LeRoy Wenstrom, Woody Wenstrom, and Ron Yannone.