THE MATHEMATICAL ASSOCIATION OF AMERICA
AMERICAN MATHEMATICS COMPETITIONS

27th Annual

AMERICAN INVITATIONAL MATHEMATICS EXAMINATION
(AIME I)

SOLUTIONS PAMPHLET

Tuesday, March 17, 2009

This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution.

Correspondence about the problems and solutions for this AIME and orders for any of the publications listed below should be addressed to:
American Mathematics Competitions
University of Nebraska, P.O. Box 81606
Lincoln, NE 68501-1606
Phone: 402-472-2257; Fax: 402-472-6087; email: amcinfo@maa.org

The problems and solutions for this AIME were prepared by the MAA's Committee on the AIME under the direction of:
Steve Blasberg, AIME Chair
San Jose, CA 95129 USA

1. (Answer: 840)

   For a 3-digit sequence to be geometric, there are numbers a and r such that the terms of the sequence are a, ar, ar^2. The largest geometric number must have $a \le 9$. Because both ar and ar^2 must be digits less than 9, r must be a fraction less than 1 with a denominator whose square divides a. For $a = 9$, the largest such fraction is $\frac{2}{3}$, and so the largest geometric number is 964. The smallest geometric number must have $a \ge 1$. Because both ar and ar^2 must be digits greater than 1, r must be at least 2 and so the smallest geometric number is 124. Thus the required difference is $964 - 124 = 840$.

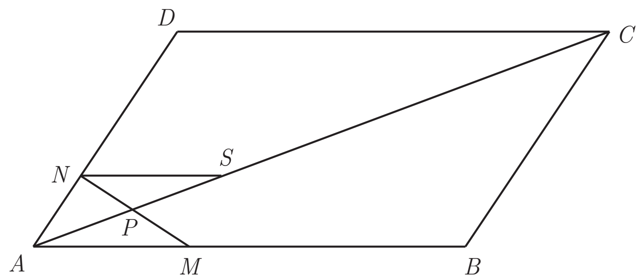
2. (Answer: 697)

   Let $z = a + bi$. Then $z = a + bi = (z + n)4i = -4b + 4i(a + n)$. Thus $a = -4b$ and $b = 4(a + n) = 4(n - 4b)$. Solving the last equation for n yields $n = \dfrac{b}{4} + 4b = \dfrac{164}{4} + 4 \cdot 164$, so $n = 697$.

3. (Answer: 011)

   The conditions of the problem imply that $\binom{8}{3}p^3(1-p)^5 = \frac{1}{25}\binom{8}{5}p^5(1-p)^3$, and hence $(1-p)^2 = \frac{1}{25}p^2$, so that $1-p = \frac{1}{5}p$. Thus $p = \frac{5}{6}$, and $m+n = 11$.

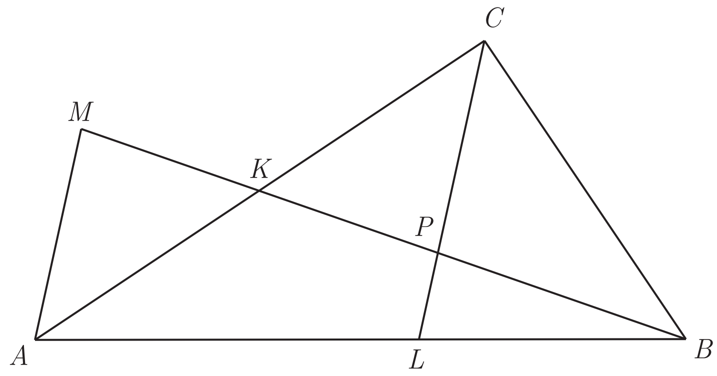
4. (Answer: 177)



   Let point S be on $\overline{AC}$ such that $\overline{NS}$ is parallel to $\overline{AB}$. Because $\triangle ASN$ is similar to $\triangle ACD$, $\dfrac{AS}{AC} = \dfrac{AP + PS}{AC} = \dfrac{AN}{AD} = \dfrac{17}{2009}$. Because $\triangle PSN$ is similar to $\triangle PAM$, $\dfrac{PS}{AP} = \dfrac{SN}{AM} = \dfrac{\frac{17}{2009}CD}{\frac{17}{1000}AB} = \dfrac{1000}{2009}$, and so $\dfrac{PS}{AP} + 1 = \dfrac{3009}{2009}$. Hence $\dfrac{\frac{17}{2009}AC}{AP} = \dfrac{3009}{2009}$, and $\dfrac{AC}{AP} = 177$.

5. (Answer: 072)

Because the diagonals of $APCM$ bisect each other, $APCM$ is a parallelogram. Thus $\overline{AM}$ is parallel to $\overline{CP}$. Because $\triangle ABM$ is similar to $\triangle LBP$, $\frac{AM}{LP} = \frac{AB}{BL} = 1 + \frac{AL}{BL}$. Apply the Angle Bisector Theorem in triangle ABC to obtain $\frac{AL}{BL} = \frac{AC}{BC}$. Therefore $\frac{AM}{LP} = 1 + \frac{AC}{BC}$, and $LP = \frac{AM \cdot BC}{AC+BC}$. Thus $LP = \dfrac{180 \cdot 300}{450 + 300} = 72$.



6. (Answer: 412)

For a positive integer k, consider the problem of counting the number of integers N such that $x^{\lfloor x \rfloor} = N$ has a solution with $\lfloor x \rfloor = k$. Then $x = \sqrt[k]{N}$, and because $k \le x < k + 1$, it follows that $k^k \le x^k \le (k+1)^k - 1$. Thus there are $(k+1)^k - k^k$ possible integer values of N for which the equation $x^k = N$ has a solution. Because $5^4 < 1000$ and $5^5 > 1000$, the desired number of values of N is $\displaystyle\sum_{k=1}^{4}[(k+1)^k - k^k] = 1 + 5 + 37 + 369 = 412$.

7. (Answer: 041)

Rearranging the given equation and taking the logarithm base 5 of both sides yields

$$a_{n+1} - a_n = \log_5(3n + 5) - \log_5(3n + 2).$$

Successively substituting $n = 1, 2, 3, \ldots$ and adding the resulting equations produces $a_{n+1} - 1 = \log_5(3n+5) - 1$. Thus the closed form for the sequence is $a_n = \log_5(3n + 2)$, which is an integer only when $3n + 2$ is a positive integer power of 5. The least positive integer power of 5 greater than 1 of the form $3k + 2$ is $5^3 = 125 = 3 \cdot 41 + 2$, so $k = 41$.

8. (Answer: 398)

For $n \geq 1$, let T_n denote the sum of all positive differences of all pairs of elements of the set $\{2^0, 2^1, 2^2, \ldots, 2^n\}$. Given two elements in this set, if neither equals 2^n, then the difference of these elements contributes to the sum T_{n-1}. Thus

$$\begin{aligned} T_n &= T_{n-1} + \left(2^n - 2^{n-1}\right) + \left(2^n - 2^{n-2}\right) + \cdots + \left(2^n - 2^0\right) \\ &= T_{n-1} + n \cdot 2^n - (2^n - 1). \end{aligned}$$

By applying this recursion repeatedly, it follows that

$$\begin{aligned} T_n &= \sum_{k=1}^{n} (k \cdot 2^k - 2^k + 1) \\ &= \sum_{k=1}^{n} k \cdot 2^k - \sum_{k=1}^{n} 2^k + \sum_{k=1}^{n} 1 \\ &= \left(\sum_{k=1}^{n} k \cdot 2^k \right) - (2^{n+1} - 2) + n \\ &= \left(\sum_{k=1}^{n} \left(\sum_{i=k}^{n} 2^i \right) \right) - (2^{n+1} - 2) + n \\ &= \left(\sum_{k=1}^{n} \frac{2^k (2^{n-k+1} - 1)}{2 - 1} \right) - (2^{n+1} - 2) + n \\ &= \left(\sum_{k=1}^{n} (2^{n+1} - 2^k) \right) - (2^{n+1} - 2) + n \\ &= n \cdot 2^{n+1} - (2^{n+1} - 2) - (2^{n+1} - 2) + n \\ &= (n-2)2^{n+1} + n + 4. \end{aligned}$$

Setting $n = 10$ gives $T_{10} = 2^{14} + 14 = 16398$. Thus the required remainder is 398.

9. (Answer: 420)

The number of possible orderings of the given seven digits is $\dfrac{7!}{4!\,3!} = 35$. These 35 orderings correspond to 35 seven-digit numbers, and the digits of each number can be subdivided to represent a unique combination of guesses for A, B, and C. Thus, for a given ordering, the number of guesses it represents is the number of ways to subdivide the seven-digit number into three nonempty sequences, each with no more than four digits. These subdivisions have possible lengths $1|2|4$, $2|2|3$, $1|3|3$, and their permutations. The first subdivision can be ordered in 6 ways and the second and third in 3 ways each, for a total of 12 possible subdivisions. Thus the total number of guesses is $35 \cdot 12$ or 420.

10. (Answer: 346)

Each acceptable seating arrangement can be specified in two steps. The first step is to assign a planet to each chair according to the committee rules. The second step is to assign an individual from the appropriate planet to each seat. Because the committee members from each planet can be seated in any of 5! ways, the second step can be completed in $(5!)^3$ ways. Thus N is the number of ways in which the first step can be completed.

In clockwise order around the table, every group of one or more Martians seated together must be followed by a group of one or more Venusians and then a group of one or more Earthlings. Thus the possible assignments of planets to chairs are in one-to-one correspondence with all sequences of positive integers $m_1, v_1, e_1, \ldots, m_k, v_k, e_k$ with $1 \le k \le 5$ and $m_1 + \cdots + m_k = v_1 + \cdots + v_k = e_1 + \cdots + e_k = 5$. For each k, the number of ordered k-tuples $(m_1, \ldots, m_k)$ with $m_1 + \cdots + m_k = 5$ is $\binom{4}{k-1}$ as are the numbers of ordered k-tuples $(v_1, \ldots, v_k)$ with $v_1 + \cdots + v_k = 5$ and $(e_1, \ldots, e_k)$ with $e_1 + \cdots + e_k = 5$. Hence the number of possible assignments of planets to chairs is

$$N = \sum_{k=1}^{5} \binom{4}{k-1}^3 = 1^3 + 4^3 + 6^3 + 4^3 + 1^3 = 346.$$

11. (Answer: 600)

First note that the distance from $(0,0)$ to the line $41x + y = 2009$ is

$$\frac{|41 \cdot 0 + 0 - 2009|}{\sqrt{41^2 + 1^2}} = \frac{2009}{29\sqrt{2}},$$

and that this distance is the altitude of any of the triangles under consideration. Thus such a triangle has integer area if and only if its base is an even multiple of $29\sqrt{2}$. There are 50 points with nonnegative integer coefficients on the given line, namely, $(0, 2009)$, $(1, 1968)$, $(2, 1927)$,..., $(49, 0)$, and the distance between any two consecutive points is $29\sqrt{2}$. Thus a triangle has positive integer area if and only if the base contains 3, 5, 7,..., or 49 of these points, with the two outermost points being vertices of the triangle. The number of bases with one of these possibilities is

$$48 + 46 + 44 + \cdots + 2 = \frac{24 \cdot 50}{2} = 600.$$

OR

Assume that the coordinates of P and Q are (x_0, y_0) and $(x_0 + k, y_0 - 41k)$, where x_0 and y_0 are nonnegative integers such that $41x_0 + y_0 = 2009$, and

k is a positive integer. Then the area of $\triangle OPQ$ is the absolute value of

$$\frac{1}{2}\begin{vmatrix} x_0+k & y_0-41k & 1 \\ x_0 & y_0 & 1 \\ 0 & 0 & 1 \end{vmatrix} = \left|\frac{1}{2}(x_0y_0+ky_0-x_0y_0+41kx_0)\right|$$
$$= \left|\frac{1}{2}k(41x_0+y_0)\right| = \left|\frac{1}{2}\cdot 2009k\right|.$$

Thus the area is an integer if and only if k is a positive even integer. The points P_i with coordinates $(i, 2009-41i)$, $0 \le i \le 49$, represent exactly the points with nonnegative integer coordinates that lie on the line with equation $41x+y=2009$. There are 50 such points. The pairs of points (P_i, P_j) with $j-i$ even and $j>i$ are in one-to-one correspondence with the triangles OPQ having integer area. Thus $j-i=2p$, $1 \le p \le 24$ and for each possible value of p, there are $50-2p$ pairs of points (P_i, P_j) meeting the conditions that P_i and P_j are points on $41x+y=2009$ with $j-i$ even and $j>i$. Thus the number of such pairs and the number of triangles OPQ with integer area is

$$\sum_{p=1}^{24}(50-2p) = \sum_{q=1}^{24} 2q = 2\cdot\frac{24\cdot 25}{2} = 600.$$

12. (Answer: 011)

Let E and F be the points of tangency on $\overline{AI}$ and $\overline{BI}$, respectively. Let $IE=IF=x$, $AE=AD=y$, $BD=BF=z$, $r=$ the radius of the circle ω, $CD=h$, and k be the area of triangle ABI. Then $h=\sqrt{yz}$, and so $r=\frac{1}{2}\sqrt{yz}$. Let s be the semiperimeter of $\triangle ABI$, so that $s=x+y+z$. On one hand $k=sr=\frac{1}{2}(x+y+z)\sqrt{yz}$, and on the other hand, by Heron's Formula, $k=\sqrt{(x+y+z)xyz}$. Equating these two expressions and simplifying yields $4x=x+y+z$, or $4x=x+AB$. Thus $x=\frac{AB}{3}$ and $2s=2\cdot\frac{AB}{3}+2\cdot AB=\frac{8}{3}\cdot AB$. Hence $m+n=8+3=11$.

13. (Answer: 090)

The definition gives

$$a_3(a_2+1)=a_1+2009, \quad a_4(a_3+1)=a_2+2009, \quad a_5(a_4+1)=a_3+2009.$$

Subtracting adjacent equations yields $a_3-a_1=(a_3+1)(a_4-a_2)$ and $a_4-a_2=(a_4+1)(a_5-a_3)$. Suppose that $a_3-a_1\neq 0$. Then $a_4-a_2\neq 0$, $a_5-a_3\neq 0$, and so on. Because $|a_{n+2}+1|\ge 2$, it follows that $0<|a_{n+3}-a_{n+1}|=\frac{|a_{n+2}-a_n|}{|a_{n+2}+1|}<|a_{n+2}-a_n|$, that is, $|a_3-a_1|>|a_4-a_2|>|a_5-a_3|>\cdots$, which is a contradiction. Therefore $a_{n+2}-a_n=0$ for all $n\ge 1$, which implies that all terms with an odd index are equal, and all terms with an even index are equal. Thus as long as a_1 and a_2 are integers, all the terms are integers. The definition of the sequence then implies that $a_1 = a_3 = \frac{a_1+2009}{a_2+1}$, giving $a_1a_2 = 2009 = 7^2 \cdot 41$. The minimum value of $a_1 + a_2$ occurs when $\{a_1, a_2\} = \{41, 49\}$, which has a sum of 90.

14. (Answer: 905)

For $j = 1, 2, 3, 4$, let m_j be the number of a_i's that are equal to j. Then

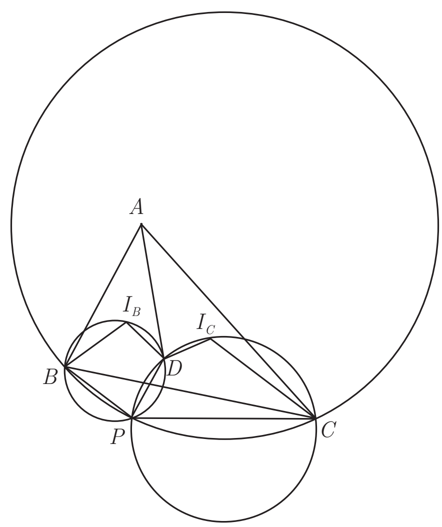
$$m_1 + m_2 + m_3 + m_4 = 350,$$
$$S_1 = m_1 + 2m_2 + 3m_3 + 4m_4 = 513, \text{ and}$$
$$S_4 = m_1 + 2^4m_2 + 3^4m_3 + 4^4m_4 = 4745.$$

Subtracting the first equation from the second, then the first from the third yields

$$m_2 + 2m_3 + 3m_4 = 163, \text{ and}$$
$$15m_2 + 80m_3 + 255m_4 = 4395.$$

Now subtracting 15 times the first of these equations from the second yields $50m_3 + 210m_4 = 1950$ or $5m_3 + 21m_4 = 195$. Thus m_4 must be a nonnegative multiple of 5, and so m_4 must be either 0 or 5. If $m_4 = 0$, then the m_j's must be $(226, 85, 39, 0)$, and if $m_4 = 5$, then the m_j's must be $(215, 112, 18, 5)$. The first set of values results in $S_2 = 1^2 \cdot 226 + 2^2 \cdot 85 + 3^2 \cdot 39 + 4^2 \cdot 0 = 917$, and the second set of values results in $S_2 = 1^2 \cdot 215 + 2^2 \cdot 112 + 3^2 \cdot 18 + 4^2 \cdot 5 = 905$. Thus the minimum value is 905.

15. (Answer: 150)



Note that

$$\angle BI_BD = \angle I_BBA + \angle BAD + \angle ADI_B = \angle BAD + \frac{\angle DBA}{2} + \frac{\angle ADB}{2}$$

and

$$\angle CI_CD = \angle I_CDA + \angle DAC + \angle ACI_C = \angle DAC + \frac{\angle CDA}{2} + \frac{\angle ACD}{2}.$$

Because $\angle BAD + \angle DAC = \angle BAC$ and $\angle ADB + \angle CDA = 180^\circ$, it follows that

$$\begin{aligned}\angle BI_BD + \angle CI_CD &= \angle BAC + \frac{\angle ACD}{2} + \frac{\angle DBA}{2} + 90^\circ \\ &= 180^\circ + \frac{\angle BAC}{2}\,.\end{aligned} \tag{1}$$

The points P and I_B must lie on opposite sides of $\overline{BC}$, and BI_BDP and CI_CDP are convex cyclic quadrilaterals. If P and I_B were on the same side, then both BI_BPD and CI_CPD would be convex. It would then follow by (1) and the fact that quadrilaterals BI_BPD and CI_CPD are cyclic that

$$\angle BPC = \angle BPD + \angle DPC = \angle BI_BD + \angle CI_CD = 180^\circ + \frac{\angle BAC}{2} > 180^\circ\,,$$

which is impossible.

By (1),

$$\begin{aligned}\angle BPC &= \angle BPD + \angle DPC = 180^\circ - \angle BI_BD + 180^\circ - \angle CI_CD \\ &= 180^\circ - \frac{\angle BAC}{2}.\end{aligned}$$

Therefore, $\angle BPC$ is constant, and so P lies on the arc of a circle passing through B and C.

The Law of Cosines yields $\cos\angle BAC = \frac{10^2+16^2-14^2}{2\cdot 10\cdot 16} = \frac{1}{2}$, and so $\angle BAC = 60^\circ$. Hence $\angle BPC = 150^\circ$, and the minor arc subtended by the chord BC measures 60°. Thus the radius of the circle is equal to $BC = 14$. The maximum area of $\triangle BPC$ occurs when $BP = PC$. Applying the Law of Cosines to $\triangle BPC$ with $BP = PC = x$ yields $14^2 = 2x^2 + x^2\sqrt{3}$, so $x^2 = \dfrac{196}{2+\sqrt{3}} = 196(2-\sqrt{3})$. The area of this triangle is $\dfrac{1}{2}x^2 \sin 150^\circ = 98 - 49\sqrt{3}$, and so $a + b + c = 150$.

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