Counting and Probability

Basic Counting:

Before we begin solving problems involving probability, we must learn some basic counting techniques.

Sometimes problems that seem the easiest are the ones that we mess up on most.

Take this problem for example:

Marianne takes a pack of index cards and numbers the cards starting with 10 and ending with 50. How many cards does she number?

If you do not understand why there are 41 cards, try this example.

Marianne takes a pack of index cards and numbers the cards starting with 00 and ending with 50. How many cards does she number?

Practice:

1. How many integers are there *BETWEEN* 500 and 600 (exclusive)?
2. How many of these integers are even?
3. Jonathan starts counting at 130 and counts by fives.
   What is the 13th number that Jonathan says?
4. Claire orders new checkbooks. Checks are numbered sequentially.
   If she orders 400 checks and the last check is number 3474, what number is on the first check?

Try to write a general rule:

How many positive integers are between a and b *exclusive* (a<b)?

How many numbers are there from a to b *inclusive* (a<b)?

Here is another easy one to mess up:

Ex. Jeremy owns a rectangular plot of land that is 60 yards long and 30 yards wide.

If he places a fence post at each corner and leaves 3 yards between each post, how many fenceposts will there be on each side?

If he places a fence post at each corner and leaves 3 yards between each post, how many fenceposts will he need to enclose his land?

Name________________________ Period _____

Basic Counting

Counting

Practice:

Remember: Whole number: 0,1,2,3,4... Integers: ... -3, -2, -1, 0, 1, 2, 3 ...

1. The soccer team has jerseys numbered from 10 to 30. Everyone on the team gets a jersey and there are three left over. How many players are on the team?

2. Chopping a carrot into slices (the usual way), how many cuts are required to make 20 pieces?

3. How many three-digit **whole numbers** are there?

4. Paul is making a ruler. He places a long mark at every **whole number**, a medium mark every half-inch, and a tiny mark every quarter-inch. How many marks will he need to make a standard six-inch ruler?

5. How many **even** perfect squares are there from 100 to 10,000 inclusive?

6. The circumference of a circular table is 30 feet. You place a set of silverware every three feet around the circumference of the table. How many place-settings are there?

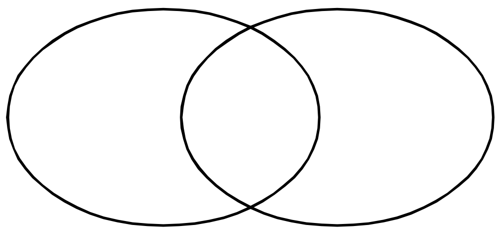
7. How many whole numbers **less than** 100 are multiples of 3 but **not** multiples of 5?

8. How many pairs of consecutive **integers** have a product less than 10,000?

Venn Diagrams

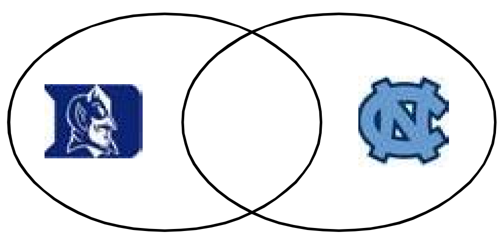
A **Venn diagram** can be used to organize counting problems where items are included or excluded:

Example:
In science classroom, 19 students have a brother, 15 students have a sister, 7 students have both a brother and a sister, and 6 students don't have any siblings at all. How many students are in the classroom?



Example:
On a survey, 40 students answered questions about their favorite sports teams. 14 students responsed that they like Duke, 18 students responded that they like UNC, and 11 students do not like Duke or UNC. How many students like both Duke and UNC?

One way to approach this problem is by drawing a Venn diagram.



These problems can also be apprached using logic:
There are 40 students, and 29 like at least one of the teams. If 14 like Duke and 18 like UNC, that makes 32 students, so 3 must like both.

Practice:
1. In a group of forty students, twenty like both pizza and hamburgers. If nine students only like pizza, and seven students only like burgers, how many students don't like pizza or burgers?

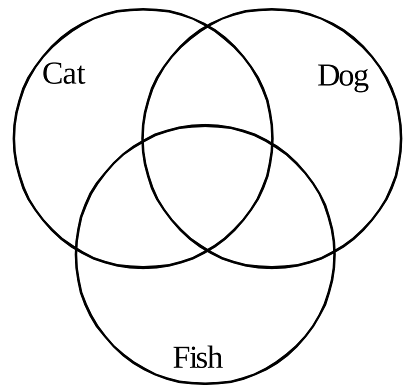
2. In a survey of 100 travelers, 86 have traveled to Europe and 31 have traveled to Asia. How many have traveled to both Europe and Asia?

3. How many numbers from 1 to 100 are divisible by two and/or three?

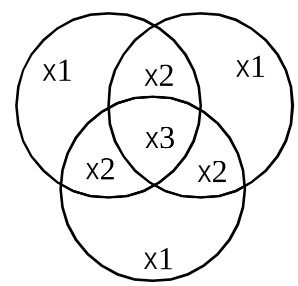
Venn Diagrams

Using more than two variables can be more difficult in a Venn diagram.

Example:
Every student who appled for admission to a veterinary school has at least one pet: 30 have a cat, 28 have a dog and 26 have fish. If 13 students have fish and a cat, 15 students have fish and a dog, 11 students have both a cat and a dog, and 4 students have a cat, a dog, and fish. How many students applied to veterinary school? Begin at the center of the diagram below and work your way out.



Using a Venn Diagram is easy enough, but you can also use the following logic:
30 have cats + 28 have a dog + 26 have fish = 84 owners.
People who own two (or more) different animals were counted multiple times, so we subtract them: 84 - 13 - 15 - 11 = 45.
This seems right, but it is different from the answer we got with the diagram. Notice that when we subtracted people with two pets, we subtracted the people who have all three pets three times (we actually wanted to subtract them twice).
Add 4 back to get 49.



Example:
7 out of 8 dentists recommend brushing your teeth after every meal, and 6 out of 8 dentists recommend flossing after every meal, and 5 out of 8 dentists recommend chewing sugarless gum after every meal, and every dentist recommends you do at least one of these three things. What is the fewest possible number of dentists who recommend doing all three?

Hard Practice:

1. How many integers from 1 to 60 are multiples of 3, 4, or 5?

2. A small auto dealership sells expensive foreign automobiles. You are looking for a black convertible Porsche. The lot has 50 cars that meet at least one of your criteria. They have 18 Porsches, 25 black cars, and 16 convertibles. There are 3 black convertibles, 4 black Porsches, and 5 convertible Porsches. How many black convertible Porsches are there?

Name________________________ Period _____

Venn Diagrams and Counting

Counting

Practice: (Calculator o.k., think hard about these and compare answers.)

1. At the pound there are 40 dogs. If 22 dogs have spots and 30 dogs have short hair, what is the fewest number of dogs that can have short hair and spots?

1.________

2. Ten friends go out to dinner together: 7 order an appetizer, 5 order a soup, and 4 order a salad. If everyone orders something, but no one orders exactly two things, how many people order all three things?

2.________

3. How many of the first 729 (27^2) counting numbers have integer square roots or cube roots?

3.________

4. All squares are both rectangles and rhombuses. All rhombuses and rect-angles are parallelograms. On a sheet of paper Joshua draws 19 rect-angles, 15 rhombuses, and 7 squares. How many parallelograms did Joshua draw?

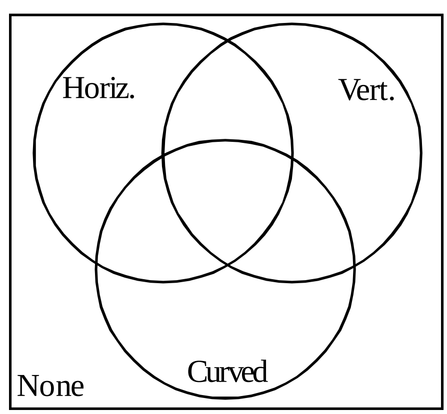
4.________

5. How many of the smallest 1,000 integers are divisible by 5, 6, and/or 7?

5.________

6. In the alphabet below, letters have vertical lines, horizontal lines, and curved lines (some have more than one property). Create a Venn diagram which places the correct number of letters in each category. Remember, there should be just 26 letters in your diagram.

ABCDEFGHIJKLMNOPQRSTUVWXYZ

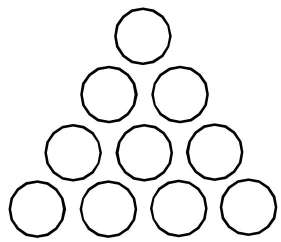


7. If 60% of the students are males, 50% of the students have dark hair, 35% are short, and just 5%are short dark-haired males, what is the largest percent of the student population that can be tall, dark-haired males?

7.________

The Bowling Pin Pattern.

How many bowling pins are set up in a standard bowling lane?
Try to draw the pattern of pins (there are four rows).



If a fifth row is added, how many pins will there be?

What if there were 100 rows of pins!?
Lets look for a pattern to see if we can solve this more easily. There are several ways to explain how to count the number of pins in *n* rows of the bowling pin pattern.

Method 1: Find the average...

Lets solve for twelve rows. You should see that there will be:
1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 pins.

The mean of all of these numbers is the same as the median (the number right in the middle). Finding the mean can be done by adding the first and the last numbers and dividing by 2.

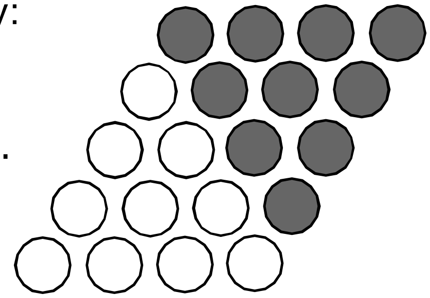
The average for 12 numbers is $\frac{1+12}{2}=6.5$.

If we multiply this by the number of terms (12), we get: $12\cdot\frac{1+12}{2}=12(6.5)=78$

The general formula for the sum of the first n positive integers is $\frac{n(n+1)}{2}$.

Method 2: Think of it geometrically:

See if you can figure out how this relates to the general formula above.



The Bowling Pin Pattern Part 2

Method 3: The handshake problem.
12 strangers meet to go bowling. If everyone shakes everyone else's hand exactly once, how many handshakes have occured?

Lets look this scenario in two ways:

A. Lets name one of the strangers Fred.
How many hands does Fred have to shake?
Lets call the second stranger Phyllis.
How many hands does Phyllis have to shake
(remember, she already shook Fred's hand.
If you see where I am going with this, finish the problem.

B. Everyone in a room of 12 people has to participate in
how many handshakes (every person shakes how many
other people's hand?)
Is Fred shaking Phyllis' hand the same handshake as when
Phyllis shakes Fred's hand? How do we eliminate the overcounts?
How many handshakes occured?

Try to write the general formula for the handshake problem where n people shake hands. It is a little different from the bowling pin formula, but the math relies on the same principle.

***Important! You should not need to memorize these formulas once you understand them.**

More examples:

1. Draw five non-colinear points on a sheet of paper(no three points should be on the same straight line). How many lines does it take to connect every point to every other point? If there were 30 points drawn, how many segments would be needed to connect them all?

2. A volleyball league has 16 teams. In the regular season, every team plays every other team exactly once. How many games are played altogether?

Name________________________ Period _____

Practice: The 'Bowling Pin' Pattern

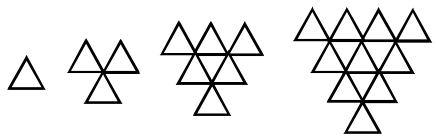
Counting

Practice:

1. Mark is stacking cans at the paint store. He starts with a row of 20 paint cans and adds rows of 19, 18, 17... until there is just one can atop the giant stack. How many cans will Mark need to complete his stack?

1.________

2. The first figure below is made of three toothpicks, each one unit long. The second figure is made from 9 of the same toothpicks. How many toothpicks will be needed to create the 16th figure?



....

2.________

3. Find the sum of each pattern below using what we have learned. You will need to find a way to modify the general formula to make it work for b and c.

a. 1 + 2 + 3 + 4 + 5 + ... + 49 + 50

3a.________

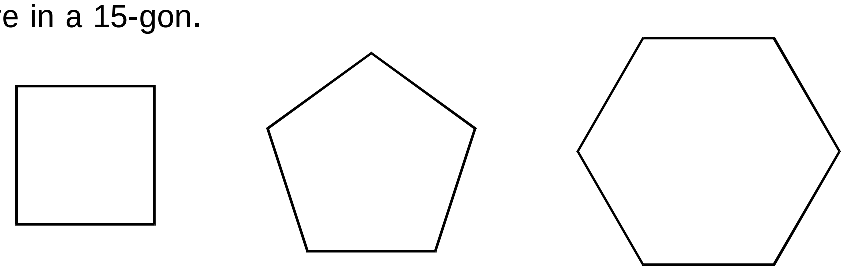
b. 2 + 4 + 6 + 8 + 10 + ... + 48 + 50

3b.________

c. 51 + 52 + 53 + 54 + ... + 99 + 100

3c.________

4. There are two diagonals in a square. A pentagon has 5 diagonals. A hexagon has 9 diagonals. Draw-in the diagonals. Look for a pattern to determine how many diagonals there are in a 15-gon.



....

4. (15-gon)________

5. Several couples arrive at a dinner party. Each person at the party shakes the hand of every other person (not including his or her spouse). If there were a total of 112 handshakes, how many couples attended the party?

5.________

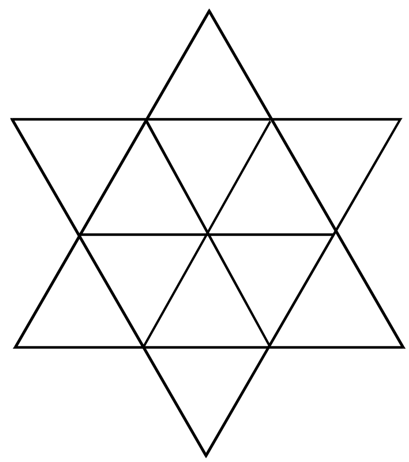
6. What is the largest prime factor of 1 + 2 + 3 + 4 + 5 + ... + 60?

6.________

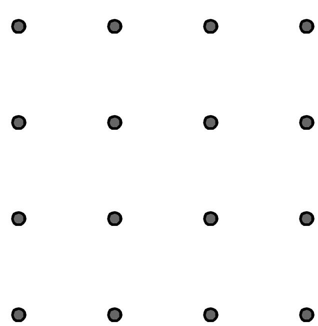
Keeping Organized: Casework

Sometimes the most difficult part of counting is keeping organized. This is particularly true when working with diagrams.

Example:
How many triangles (of any size) are there in the figure below?



Example:
How many squares (of any size) can be created by connecting four dots on the grid below?
hint: There are five different sizes!



Name_________________________ Period _____

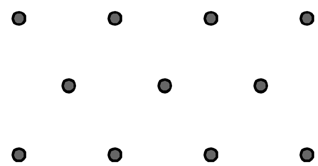
Practice: Casework

Counting

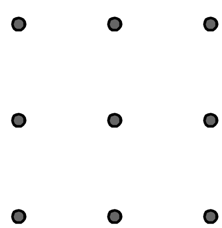
Practice:

For each diagram, assume that the grid is regular (the distances are all congruent between adjacent points and the angles are as they appear.

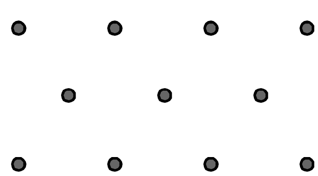
1. How many isosceles (this includes equilateral) triangles can be formed by connecting three points on the grid below?



2. How many right triangles can be formed by connecting three points on the grid below?



3. How many parallelograms (this includes rectangles and rhombuses) can be formed by connecting four points on the grid below?



Name________________________ Period _____

Fundamental Counting Principle

Counting

Counting Outcomes:

Dice and Coins

1. **How many possible outcomes are there when:**
   One coin is flipped?
   Two coins are flipped?
   Three coins are flipped?
   Ten coins are flipped?

2. **How many possible outcomes are there when:**
   One (standard six-faced) die is rolled?
   Two dice are rolled?
   Three dice are rolled?
   Two coins are flipped and a die is rolled?

The Fundamental Counting Principle states that if there are *m* ways that one event can happen, and *n* ways that a second event can happen, then there are *mn* ways that both events can happen.

Example:

A bag contains a penny, nickel, a dime, and a quarter. A second bag contains a $1, $5, and $10 bill. If a coin and a bill are selected at random, how many different values are possible?

Example: You do not need to be counting events.

At subway you can order your meatball sub in two sizes, with or without cheese, and you get your choice of 5 different types of bread. How many different meatball subs are available?

Example:

A (standard six-faced) die is rolled three times to create a three-digit number.
How many numbers can be created using this method which are greater than 400?

Something everyone should know about dice:

With a pair of dice: How many ways are there to roll a 2? 3? 4? ... 12?

What is the most common roll with a standard pair of dice?

Name________________________ Period _____

Fundamental Counting Principle

Counting

Practice:

1. How many outcomes are possible when a coin is flipped seven times?

1.________

2. A company that makes mens' jeans offers three styles, eight waist sizes, ten inseam lengths, and two different colors. How many different pairs of mens jeans do they manufacture?

2.________

3. How many four-digit numbers can be formed using only the digits 5, 6, 7, and 8? (You may use digits more than once.)

3.________

4. How many ten digit numbers can be created using only ones and twos?

4.________

5. How many ten digit numbers can be created using only ones and zeros? (This is slightly different from #3, figure out why.)

5.________

6. A standard license plate usually consists of three letters followed by a three-digit number (000<n<1000). How many standard license plates can be created with this system if the letter O is not used?

6.______________

7. Pizzas at Mario's come in three sizes, and you have your choice of 9 toppings. This does not create 27 possible combinations, it creates over 1,500 choices. Figure out how this problem is different and find the exact number of choices available. #1 and #4 may help you solve this problem.

7.________

8. Each of the digits 0-4 is used once to create a five-digit number. How many numbers can be created?

8.________

Name________________________ Period _____

Review

Counting

Quiz Review:
Basic Counting

1. How many numbers are there from 200-300 inclusive?

1.________

2. A field is shaped like an octagon. Fence posts surround the field, with four on each side including a post at each vertex of the octagon. How many posts surround the field?

2.________

3. A 50-foot ladder has rungs spaced 1 foot apart, starting 1 foot from each end. How many rungs are on the ladder?

3.________

Bowling pins and handshakes:

4. Solve for x if $x = 30 + 29 + 28 + 27 + \ldots 3 + 2 + 1$

4.________

5. If 99 people stand in a circle, and each person shakes the hand of the person on either side, how many handshakes take place?

5.________

6. There are 15 points on a sheet of paper numbered from 1 through 15. How many lines must be drawn to connect every pair of points whose sum is odd?

6.________

Venn Diagrams:

7. Of the doctors surveyed, 29 run daily, 35 have a gym membership, 15 run daily and are gym members, and 4 neither run nor have a gym membership. How many doctors were surveyed?

7.________

8. How many positive integers less than 1,000 are even and/or divisible by 3?

8.________

Name________________________ Period _____

Review

Counting

Quiz Review:

Fundamental Counting Principle:

9. How many outcomes are possible if one coin is flipped and one die is rolled?

9.________

10. To use the copier, teachers must enter three initials (first, middle, last name) on a keypad. How many combinations of three initials are possible?

10.________

11. How many five-digit positive integers use only the digits 0 through 3 (digits may repeat)?

11.________

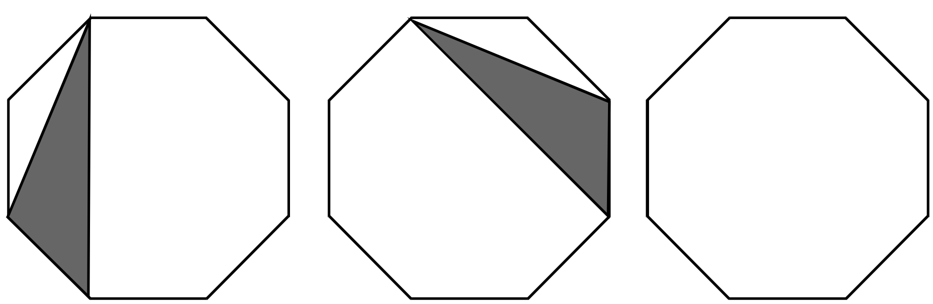
Casework:

12. Each of the six faces of a cube is painted either black or white. Two cubes are considered identical if one cube can be rotated to look like another (for example, there is only one distinct cube that has exactly one white face. How many distinct cubes are possible?

12.________

13. How many *distinct* triangles can be formed by connecting three of the vertices of a regular octagon. Two triangles are considered distinct if they cannot be rotated or refkected to appear congruent. For example, the two triangles drawn below are not distinct.

13.________



Name________________________ Period _____

Practice Quiz: Counting

Counting

Solve each:

1. Andrew collects baseball cards. He has the complete set of numbered cards starting with #100 and ending with #200. How many cards is this?

1.________

2. What is the sum of the first 49 positive integers?
(1 + 2 + 3 + 4 ... + 48 + 49)

2.________

3. If you are organizing a basketball league with five teams, and each team must play every other team exactly once, how many games will be played altogether?

3.________

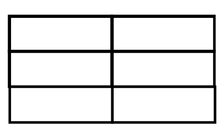
4. How many *two-digit* positive integers are even?

4.________

5. A straight fence is 150 feet long. Fenceposts are spaced 5 feet apart, with one post at each end of the fence. How many posts are there?

5.________

6. How many rectangles (of any size) are there in the diagram below?



6.________

7. Bryan cuts 30 triangles out of a sheet of paper. 22 of the triangles are isosceles, and 25 are right triangles. How many are isosceles right triangles?

7.________

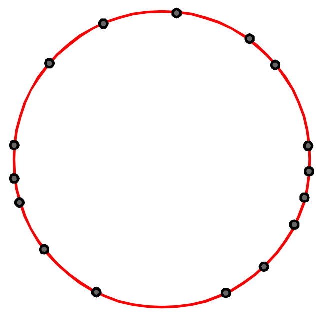
Name________________________ Period _____

Practice Quiz: Counting

Counting

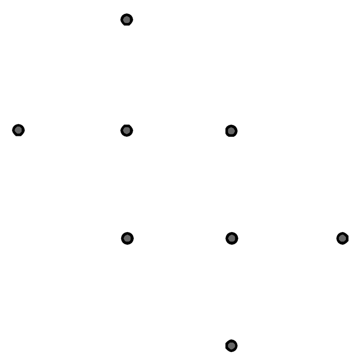
Solve each:

8. Sixteen points are placed on the circumference of a circle.
How many lines will it take to connect every point to every other point?



8._______

9. How many right triangles can be created by connecting points
on the unit grid below?



9._______

10. Using each of the digits 1 through 5 exactly once, how many 5-digit numbers can be created which are divisible by 4? (The divisibility rule for 4 is: The number created by the last two digits must be divisible by 4).

10._______

Name________________________ Period _____

Quiz: Counting

Counting

Solve each:
You are not expected to be able to answer all of the questions on this quiz.

1. Chapter 11 starts on page 120 and finishes on page 145.
How many pages are there in Chapter 11?

1.________

2. In a stack of tomato cans, each row has one less can than the row below it until there is just one can on top. If the stack has 12 cans on the bottom row, how many cans are there in the stack altogether?

2.________

3. Ten people all go to a math conference. If every person shakes hands with every other person exactly once, how many handshakes occur?

3.________

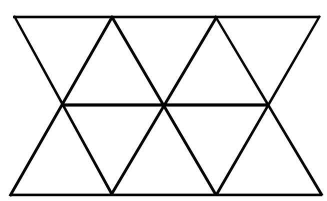
4. How many multiples of three are between 20 and 80?

4.________

5. For homework, Jessica needs to complete problems 10 through 36, and odd problems from 41 to 51. How many problems is this altogether?

5.________

6. How many equilateral triangles (of any size) are there in the diagram below?



6.________

7. What is the sum of all the odd positive integers which are less than 100?
(1 + 3 + 5 + ... + 97 + 99)

7.________

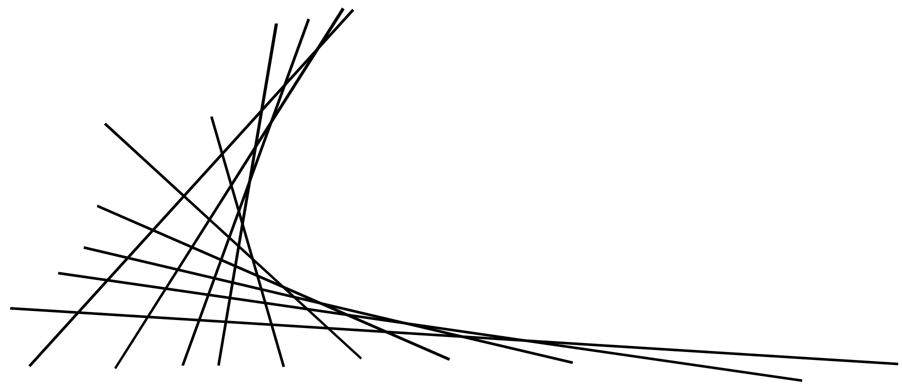
Name________________________ Period _____

Quiz: Counting

Counting

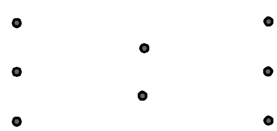
Solve each:

8. Twenty distinct straight lines are drawn on a page so that each line intersects every other line, but no more than two lines intersect at any point. How many points of intersection are created?
The diagram below has just ten lines.



8.__________

9. How many different isosceles triangles can be formed by connecting three of the dots below?



9.__________

10. At the salad bar, you can choose between three types of lettuce, six salad dressings, and there are ten toppings that you can put on your salad. How many combinations of lettuce, dressing, and toppings are available?

10.__________

Factorials

You may have seen an exclamation point in a math problem at some point and wondered, “What is so exciting about that number?”

No, 4! does not mean **“FOUR!!!”**

That exclamation point is actually not there to represent excitement or volume, it is **factorial notation.**

n! is the product of all positive integers less than or equal to n.

Examples make this simple:

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$$

$$10! = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 3{,}628{,}800$$

$$15! = 15 \cdot 14 \cdot \ldots \cdot 3 \cdot 2 \cdot 1 = 1{,}307{,}674{,}368{,}000$$

You can see that factorials can get big fast! You will not be asked to simply compute large factorials, but there are many interesting problems involving factorials that we will review.

First, some simple arithmetic and Algebra:

Examples:

1. $\dfrac{5!}{4!} =$ **2.** $\dfrac{5!}{10} =$ **3.** $\dfrac{5 \cdot 4!}{5!} =$

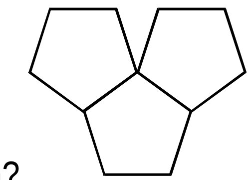
Harder Practice Problems:

1. $\dfrac{21!}{19!} =$ **2.** $\dfrac{6!}{60} =$ **3.** $\dfrac{5! + 4!}{3!} =$

Permutations

Consider the following:
Five students are running a race. In how many ways can the five students place 1st, 2nd, and 3rd?

An organization is choosing colors for the three pentagons in its logo. They have narrowed their choice to 7 colors, and want to have three diffierent colors in their logo. How many possible logos can be created by choosing three of the seven colors and using one for each pentagon?



There is a formula for **permutations** and some notation which we must consider. **P(n,r)** means the number of permutations of n things taken r at a time.

For example, five people in a race taking three places is **P(5,3).**

$$P(5,3) = 5 \cdot 4 \cdot 3$$

The formula for **P(n,r).**

$$P(n,r) = \frac{n!}{(n-r)!}$$

Got that? Now, you can really forget this... use common sense, don't memorize a formula. You should, however, remember what **P(7,2)** means.

Practice:
1. Find **P(10,3)**.
2. Find **P(8,6)**.
3. For the pick 3 lottery, six balls numbered 1 through 6 are placed in a hopper and randomly selected one at a time without replacement to create a three-digit number. How many different three-digit numbers can be created?
4. How many ways can five books be ordered on a shelf from left to right?
5. Eight people are asked to select a leadership team: president, vice-president, and secretary among themselves. How many different leadership teams are possible?

Permutations

Letter Arrangements:

Perhaps the most commonly asked questions involving permutations involve arrangements of letters (probably because these problems are easy to write).

Ex.

How many different four-letter 'words' can be formed by rearranging the letters in the word MATH?

Solution: There are four letters to choose from for the first spot, leaving three for the second, two for the third, and one remaining for the end of the word:

$$P(4,4) = 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$$

Ex.

How many different four-letter 'words' can be formed by rearranging the letters in the word COUNTS?

Solution: This is still just a basic permutation. There are:

$$P(6,4) = 6 \cdot 5 \cdot 4 \cdot 3 = 360$$ 'words' this time.

Ex.

How many different seven-letter 'words' can be formed by rearranging the letters in the word ALGEBRA?

Solution: This is more difficult. Notice that there are two A's. No need to panic, just pretend for a moment that the A's are different. We'll call one of the them A_1 and the other A_2.

$$P(7,7) = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5{,}040$$ 'words'.

However, we overcounted. $A_1LGEBRA_2$ is the same as $A_2LGEBRA_1$, just like A_1A_2LEGBR is the same as A_2A_1LEGBR. To eliminate the extra cases, we need to divide by the ways that the As can be arranged, which in this case is just 2.

$$\frac{P(7,7)}{2} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2} = 2{,}520$$ 'words'.

Practice:

Find the number of arrangements of the letters of the word COOKBOOK.

Permutations

Restrictions:

Often there are restrictions placed on permutation problems:

Examples:

1. How many *even* five-digit numbers contain each of the digits 1 through 5?

2. How many of the arrangements of the letters in the word COUNTING contain a double-N?

3. In how many of the arrangements of the letters in the word EXAMPLE are the letters A and M adjacent to each other?

Practice:

1. How many arrangements of the letters in the word START begin with a T?

2. How many arrangements of the letters in the word BEGIN start with a vowel?

3. How many arrangements of the letters in the word BEGINNING have an N at the beginning?

4. How many of the arrangements of the digits 1 through 6 have the 1 to the left of the 2?

Practice:

1. Seven students line up on stage. If Molly insists on standing next to Katie, how many different ways are there to arrange the students on stage from left to right?

2. How many arrangements of the letters in the word ORDERED include the word RED?

3. How many arrangements of the letters in the word ORDERED begin and end with the same letter?

4. There are nine parking spots in front of the building for six teachers and the principal. If the principal always gets one of the three shady spots, how many ways can all seven cars be parked in the lot?

Name__________________________ Period _____

Practice: Letter Arrangements

Counting

Find the number of ways that the letters in each word can be arranged:

1. WORD **2.** LADDER **3.** ICICLE **4.** BOOKKEEPER

1.________ 2.________ 3.________ 4.________

5. How many arrangements of the letters in the word FOOT include a double-O?

5.________

6. How many arrangements of the letters in the word RECIPE include a double-E?

6.________

7. How many arrangements of the digits 1 through 7 create a 7-digit number that is greater than 5 million?

7.________

8. How many arrangements of the letters in the word COUNTING have the letters COUNT grouped together (in any order)?

8.________

9. How many arrangements of *5-letters* from the name BATTERSON include a double-T?

9.________

Combinations

Consider the following problems:

1. There are six scrabble letters left in the bag at the end of the game (FHJSU and Y). If you reach in and grab two letters, how many different pairs of letters are possible?

2. Kelly wants to offer three sodas at her snack stand. She has a list of 8 sodas to choose from. How many combinations of sodas are possible?

3. Roger has won a contest at the fair, and gets to choose four different prizes from a set of nine. How many **combinations** of four prizes can he **choose** from a set of nine?

The primary difference is that ORDER DOES NOT MATTER.
There is a formula for **combinations** and some notation which we must consider. **C(n,r)** means the number of combinations of n things taken r at a time.

For example, choosing four prizes from a set of nine is **C(9,4).**

$$C(9,4) = \frac{9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 3 \cdot 2 \cdot 1} = 126$$

The formula for **C(n,r).**

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

Don't memorize it, understand it!

$C(n,r)$ is also notated as $\binom{n}{r}$ or nCr and is often called 'choose'.

Practice:

1. Find $_6C_2$.

2. Find $C(6,4)$.

3. Explain why the answers above are equal.

4. How many ways can three books be chosen from a shelf of 20?

5. Eight people are asked to select a leadership team of three members. How many different leadership teams are possible?

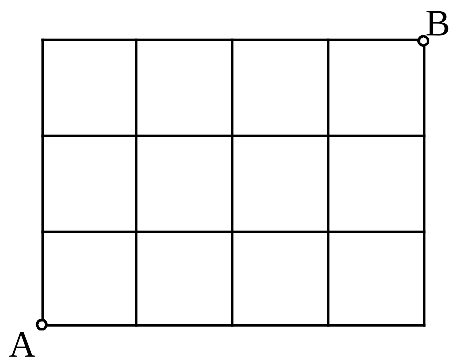
Paths and Combinations

Paths:

Tracing routes across grids is a common way to show how combination problems can arise in less obvious problems.

Ex.

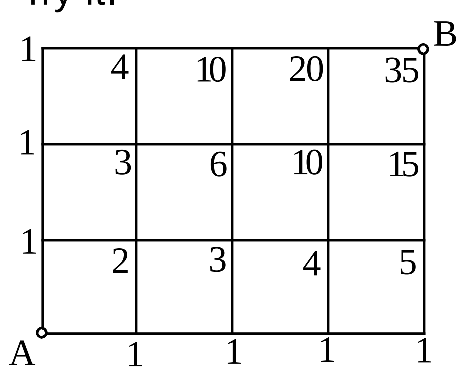
Starting from point A on the unit grid below, how many unique 7-unit paths are there from A to B?



Solution 1: Take any path and represent it as s eries of moves. Write each move as a U (up) or R (right). Every single 7-unit path consists of four Rs and three Us. The question then becomes: How many ways can we rearrange the letters RRRRUUU:

$$C(7,4)=\frac{7\cdot 6\cdot 5\cdot 4}{4\cdot 3\cdot 2\cdot 1}=35$$

Solution 2: We can also show this by counting the number of ways there are to get to each point. Try it.



Practice: James has to drive 7 blocks north and 3 blocks west to get to work on a city grid of two-way streets. How many different ways can he drive 10 blocks to work?

Name________________________ Period _____

Combinations and Permutations

Counting

Solve each:

1. How many ways can six songs be placed in order on a CD?

1.________

2. How many different arrangements of the letters in the word COUNTING are possible?

2.________

3. Of the 13 players on a soccer team, 11 are starters. How many different teams of 11 starters are possible?

3.________

4. Ten students rush into the cafeteria and take six seats at a table. How many possible combinations of students are left standing?

4.________

5. A phone number has seven digits and cannot start with a 0 or a 1.
How many phone numbers can be created if digits cannot be repeated?
(hint: How many choices are there for the first digit? the second? etc.)

5.________

6. Jacob has two red pegs to place in the game board below. How many different placements are possible?

o o o o o o
1 2 3 4 5 6

6.________

7. Jacob has three red pegs and three blue pegs to fill six holes in a game board. How many different color combinations are possible?

o o o o o o
1 2 3 4 5 6

7.________

8. Jacob has two red pegs, two green pegs, and two blue pegs to fill six holes in a game board. How many different color combinations are possible?

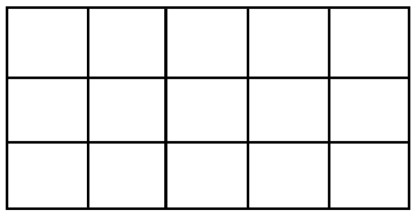
o o o o o o
1 2 3 4 5 6

8.________

Beyond Casework

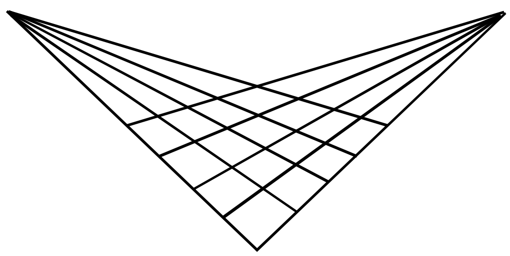
Consider the following:

How many distinct rectangles can be formed by tracing lines on the grid below?

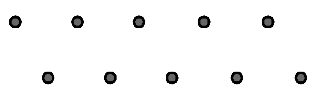


To create a rectangle, you must choose two of the six vertical lines, and two of the four horizontal lines.

Next, how many triangles are there in the figure below?



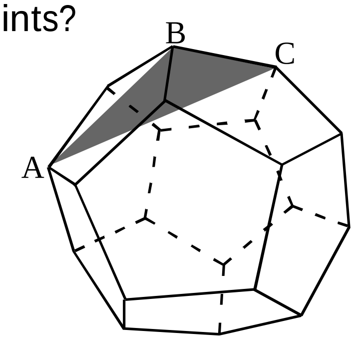
Finally, how many distinct triangles can be created by connecting three of the points below?



Practice:

1. Six points are placed on the cricumference of a circle. How many distinct triangles can be formed by connecting three of the points?

2. A regular dodecahedron has 12 pentagonal faces, 20 vertices, and 30 edges. How many triangles can be formed by connecting three vertices of a regular dodecahedron? (ex. $\triangle$ABC shown)



Name_________________________ Period _____

Beyond Casework

Counting

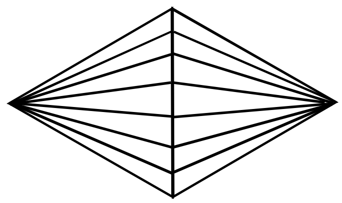
Solve each:

1. How many distinct rectangles (including squares) can be formed by tracing the gridlines on a standard chessboard (8x8)?

1.________

2. How many distinct triangles are there in the figure below?

2.________



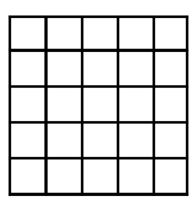
3. How many distinct triangles can be formed by connecting three vertices of a cube?

3.________

4. Ten points (A through J) are placed on the circumferences of a circle, and each is connected to all of the other points. How many of the triangles that have three points on the circumference of the circle include point F?

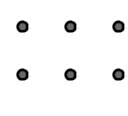
4.________

5. How many rectangles (including squares) can be created by tracing the lines of the 5x5 grid below which have a perimeter *greater* than 6 units?



5.________

6. How many distinct triangles can be formed which use three of the points on a 3x3 grid as vertices?



6.________

Complementary Counting

Consider the following:

1. Paul flips a fair coin eight times. In how many ways can he flip at least two heads?

It is possible to figure out how many ways Paul can flip two, three, four, five, six, seven, or eight heads: $\binom{8}{2}+\binom{8}{3}+\binom{8}{4}+\binom{8}{5}+\binom{8}{6}+\binom{8}{7}+\binom{8}{8}=247$

It is much easier to figure out the total number of outcomes (2^8) and subtract the number of ways he can flip 0 or 1 heads (these should be easy to do in your head). $2^8-9=256-8-1=247$

2. Corey and Tony are friends on a basketball team. There are eight players on the team. How many starting lineups (of five players) include Corey, Tony, or both?

Again, it requires a bit of computation to figure out how many teams include just Corey, just Tony, and both. Instead, consider how many starting lineups are possible without restrictions (choosing 5 from a group of 8: 8C5) and then subtract the ones that do not include Corey or Tony (choosing 5 from the remaining 6 players: 6C5).

$$\binom{8}{5}-\binom{6}{5}=56-6=50$$

We are really trying to count what we don't want, which is often easier to find and subtract from the total.

31

Practice:

1. How many two-digit integers use two different digits?

2. How many positive integers less than 1,000 are not perfect squares?

3. There are seven parking spaces in a row which must be assigned to four co-workers. How many ways can the spaces be assigned if at least two of the assigned spaces must be next to each other?

4. How many different ways can six friends stand in line at the movies if Alice and David refuse to stand next to each other?

Name________________________ Period _____

Complementary Counting

Counting

Challenge Problems. Solve each:

1. How many ways can a pair of dice be rolled so that the product of the two numbers rolled is greater than 5?

1.________

2. How many positive integers less than 50 are not divisible by 7 or 11?

2.________

3. You are playing Monopoly, and if you roll a 9, 10, or 12 with a pair of dice you will land on your opponent's green monopoly with hotels and go bankrupt. How many ways are there to avoid bankruptcy on your next roll?

3.________

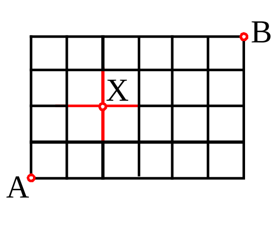
4. A fair coin is flipped ten times. How many ways are there to flip *more* heads than tails?

4.________

5. A lattice point is a point on the coordinate plane with integer coordinates such as (2, 3). How many ways can three lattice points be chosen to form a triangle if both coordinates of each lattice point are positive integers less than 5? (Source: AoPS)

5.________

6. How many ten-unit paths are there between A and B which do not pass through X?



6.________

Hints for #4,#5, and #6 on back.

Complementary Counting Hints

Hints:

4. A fair coin is flipped ten times. How many ways are there to flip *more* heads than tails?
hint: How many ways can you flip exactly the same number of heads and tails?

5. A lattice point is a point on the coordinate plane with integer coordinates such as (2, 3). How many ways can three lattice points be chosen to form a triangle if both coordinates of each lattice point are positive integers less than 5? (Source: AoPS)
hint: The lattice points form a 4x4 grid of 16 points from (1,1) to (4,4).
How many sets of three points in this set will *not* form a triangle?

6. How many ten-unit paths are there between A and B which do not pass through X?
hint: How many paths are there from A to B without restrictions?
How many ways can you get from A to X? X to B? A to B through X?
Or... you could just use the numbering method for each vertex.

Name_________________________ Period ______

Practice Quiz: Combinations etc.

Counting

Solve each:

1. How many ways can gold, silver, and bronze be awarded to the eight finalists in a dog show?

1.________

2. Solve for x: $6x = \frac{15!}{13!}$

2.________

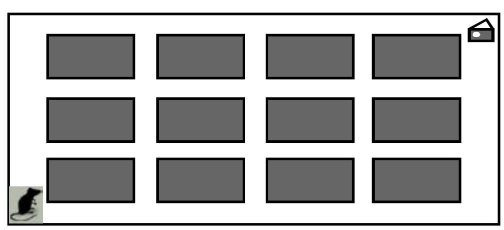
3. How many arrangements of the letters in the word PRACTICE are possible?

3.________

4. Standard Wyoming license plates consist of three different letters followed by three different numbers and do not include the letter O or digit 0. How many license plates can be made which begin with the letter A?

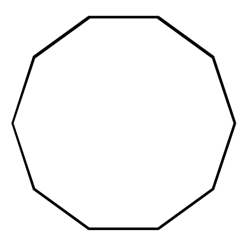
4.________

5. A mouse is placed as shown in the maze below. If the mouse only moves up or right, how many different paths can the mouse use take to get to the cheese at the end of the maze?



5.________

6. How many distinct triangles can be drawn by connecting the vertices of a regular decagon (ten sides)?



6.________

7. How many arrangements of the letters in the word MISSISSIPPI are possible?

7.________

Name________________________ Period _____

Practice Quiz: Combinations etc.

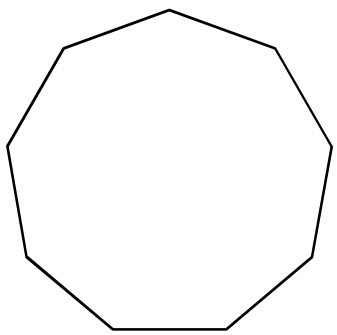
Counting

Solve each:

8. Six students run a race, including Thomas and Raj. How many ways can the six students finish first through sixth if Raj finishes ahead of Thomas (but not necessarily immediately ahead of Thomas).

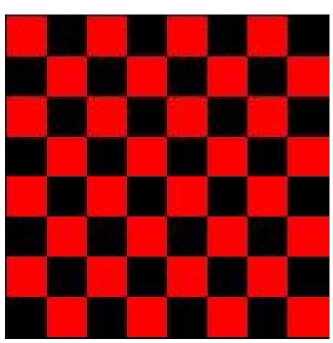
8.________

9. How many distinct *scalene* triangles (all three sides have different lengths) can be formed by connecting three vertices of a regular nonagon (nine sides)?



9.________

10. A standard checkerboard has 64 squares alternating black and red on an 8 by 8 grid. How many of the 1,296 rectangles on a checkerboard contain more than one red square?



10.________

Name_________________________ Period _____

Quiz: Combinations etc.

Counting

Solve each:

1. How many three-digit numbers use only odd digits, with no digit repeating?

1.________

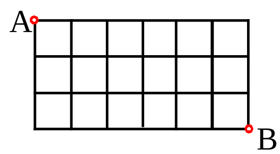
2. Homerooms are participating in a 3-on-3 basketball tournament. How many different three-person teams can a homeroom of 15 students select to participate in the tournament?

2.________

3. Ho many arrangements of the letters in the word QUIZ are possible, if the U must come immediately after the Q?

3.________

4. How many nine-unit paths are there from A to B on the grid below?



4.________

5. Three standard six-sided dice are rolled: one red, one green, and one white. How many outcomes are possible where the number rolled on the red die is greater than the number rolled on the green die, and the number on the green die is greater than the number on the white die?

5.________

6. Zip codes in the United States are five-digits long, followed by a four-digit code, for example: 27513-8046. In North Carolina, every zip code begins with either 27 or 28. How many 9-digit zip codes are possible in North Carolina if each digit can only be used once?

6.________

7. How many arrangements of the letters in the word COUNTING are possible?

7.________

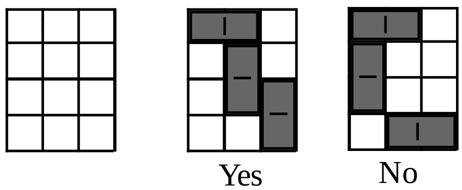
Name________________________ Period _____

Quiz: Combinations etc.

Counting

Solve each:

8. Three identical dominos are placed on the 3x4 tiled board below to create a complete pathway from the upper-left corner to the lower-right corner of the board (no dominos may touch at the corners). How many ways can this be done?

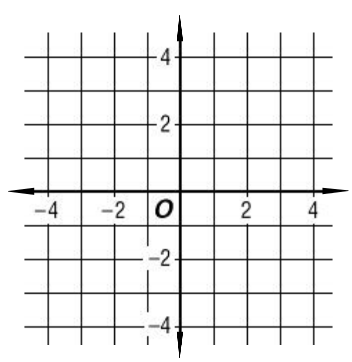
Example: 

8.________

9. An octahedron (8 sides) with sides numbered 1 through 8 die is rolled twice, and the product of the two rolls is computed. How many different rolls produce a product that is composite?

9.________

10. Lattice points in the form (x, y) are chosen such that $|x|+|y|\le 2$. How many ways can three of these lattice points be chosen which form a triangle?



10.________

Pascal’s Triangle

Perhaps you already recognize and know how to create Pascal’s Triangle. Begin at the top with a 1. Each row begins with a 1, and each number is the sum of the two above it:

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

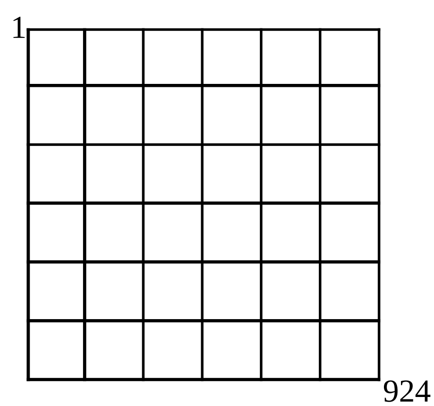
1 5 10 10 5 1

1 6 15 20 15 6 1

etc...

There is a nice relationship between combinations and Pascal’s Triangle.

Let’s begin with a problem we have seen before:
How many ways are there to get to each vertex in the diagram below by tracing the lines and using the shortest distance possible, starting from the top left and moving to the bottom right?



Example: A fair coin is flipped seven times.
How many outcomes show 0, 1, 2, 3, 4, 5, 6, and 7 tails?
Do these numbers look familiar?

Finally, compute $_6C_3$. Where can this number be found in Pascal’s triangle? What about $_6C_5$? Try to construct Pascal’s triangle using combination notation.

Pascal's Triangle

Perhaps you already recognize and know how to create Pascal's Triangle. Begin at the top with a 1. Each row begins with a 1, and each number is the sum of the two above it:

$${}_0C_0$$
$${}_1C_0 \quad {}_1C_1$$
$${}_2C_0 \quad {}_2C_1 \quad {}_2C_2$$
$${}_3C_0 \quad {}_3C_1 \quad {}_3C_2 \quad {}_3C_3$$
$${}_4C_0 \quad {}_4C_1 \quad {}_4C_2 \quad {}_4C_3 \quad {}_4C_4$$
$${}_5C_0 \quad {}_5C_1 \quad {}_5C_2 \quad {}_5C_3 \quad {}_5C_4 \quad {}_5C_5$$
$${}_6C_0 \quad {}_6C_1 \quad {}_6C_2 \quad {}_6C_3 \quad {}_6C_4 \quad {}_6C_5 \quad {}_6C_6$$

etc…

This looks confusing at first, but once you are familiar with the concept, it is a handy tool for a variety of problems (many of which we do not have time to go over, specifically, Binomial Theorem).

Using Pascal's Triangle:

Example: A fair coin is flipped 7 times. How many ways are there to flip more than 4 tails?

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

We need to add ${}_7C_5$, ${}_7C_6$, and ${}_7C_7$. These values can be found in the 8^{th} row of Pascal's Triangle. They are the last three entries (or the first three).
This gives us 21 + 7 + 1 = 29 ways.

Practice: A teacher is writing a quiz with True/False answer choices. She writes 8 questions and wants to have a minimum of three true and three false answer choices. How many ways can she organize the eight responses on her quiz if there are at least three Ts and three Fs?

Pascal's Triangle

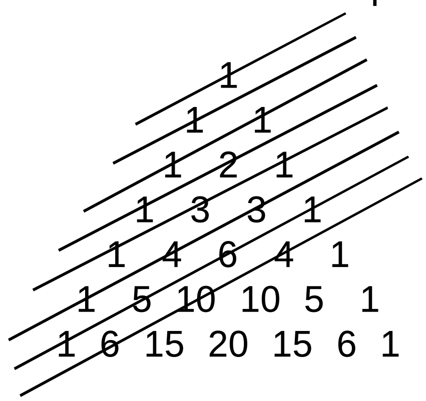
More madness with Pascal's Triangle:

Some of this stuff serves no purpose, but it is important that the sum of each row in Pascal's Triangle is a power of 2. This makes sense of course if we think of Pascal's Triangle as identifying the number of ways a coin can be flipped, 0 because there are 2^n possible outcomes for flipping a coin n times.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

Fibonacci:

I know of no use for this, but it is pretty cool. Find the sum of the numbers crossed by each diagonal line. Is this a familiar pattern of numbers?



Triangular Numbers and Perfect Squares:

Finally, look at the diagonal row that begins 1 3 6 10. These are triangular numbers. Find the sum of any pair of consecutive numbers.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

Name________________________ Period _____

Practice: Pascal's Triangle

Counting

Solve each:
You may use Pascal's Triangle where needed.

										1										
									1		1									
								1		2		1								
							1		3		3		1							
						1		4		6		4		1						
					1		5		10		10		5		1					
				1		6		15		20		15		6		1				
			1		7		21		35		35		21		7		1			
		1		8		28		56		70		56		28		8		1		
	1		9		36		84		126		126		84		36		9		1	
1		10		45		120		210		252		210		120		45		10		1

1. Find the sum of $\binom{10}{0}+\binom{10}{1}+\binom{10}{2}+\ldots\binom{10}{10}$.

1.________

2. Find the sum of $\binom{8}{0}+\binom{8}{1}+\binom{8}{2}$.

2.________

3. Jack tosses six fair coins. How many ways are there to flip an odd number of heads?

3.________

4. Jack tosses eight fair coins. How many ways are there to flip more heads than tails?

4.________

5. Find values for $a + b$ if $\binom{9}{5}+\binom{9}{6}=\binom{a}{b}$.

(hint: Think about the positions of each on Pascal's Triangle).

5.________

6. There are 22 ways to flip less than 4 heads when n coins are tossed. Find n.

6.________

7. A delivery special from the local pizza restaurant allows you to choose up to 5 toppings from a list of 9 to put on your pizza. How many combinations of toppings are available?

7.________

Name_________________________ Period _____

Practice: Pascal's Triangle

Counting

Solve each:
You may use Pascal's Triangle where needed.

										1										
									1		1									
								1		2		1								
							1		3		3		1							
						1		4		6		4		1						
					1		5		10		10		5		1					
				1		6		15		20		15		6		1				
			1		7		21		35		35		21		7		1			
		1		8		28		56		70		56		28		8		1		
	1		9		36		84		126		126		84		36		9		1	
1		10		45		120		210		252		210		120		45		10		1

8. You can choose up to four toppings for your ice cream sundae from a selection of 8 available toppings. How many combinations of toppings are available?

8.________

9. Find the pattern: $\binom{3}{2}+\binom{4}{2}=$ $\binom{8}{2}+\binom{9}{2}=$ $\binom{n}{2}+\binom{n+1}{2}=$

9.________

10. Eight coins are flipped. How many ways can the number of heads showing be a positive multiple of 3?

10.________

11. On a game show, you are offered the chance to select up to three of the ten other contestants to be on your team, but they will share the winnings (you don't have to pick anyone if you think you can do well all by yourself). How many combinations of teammates can you select?

11.________

12. How many integers *between* 1,000,000 and 2,000,000 have at least four fives?

12.________

Sticks and Stones

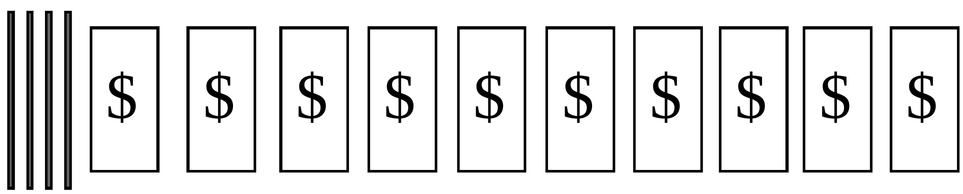
There is still one counting trick that we must cover before we begin using our counting skills to solve probability problems:

Example:
Uncle Henry has ten dollar bills to distribute among his five youngest nieces and nephews. How many ways are there to distribute the loot?

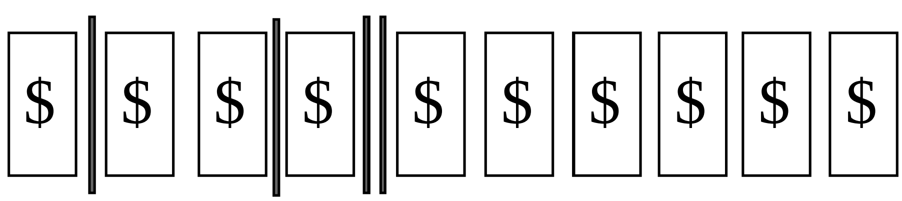
He could give all of the money to one child five ways.
He could give two dollars to each child one way.
He could give \$0.00, \$1.00, \$2.00, \$3.00, and \$4.00 5! ways.

This is getting us nowhere, there must be an easier way!

Consider the following diagram:



Henry will arrange the dividers on the left with the bills on the right. Money to the left of the first divider goes to the youngest, money between the first and second divider goes to the next child, etc. For example, in the arrangement below, the first child gets \$1.00, the second gets \$2, the third gets \$1, the fourth gets nothing, and the fifth gets \$6.



Using this model, it is simply a matter of choosing the number of arrangements of four dividers (sticks) and ten bills (stones):

$\frac{14!}{(10!)(4!)} = 1{,}001$, or more simply: $C(14,4) = 1{,}001$

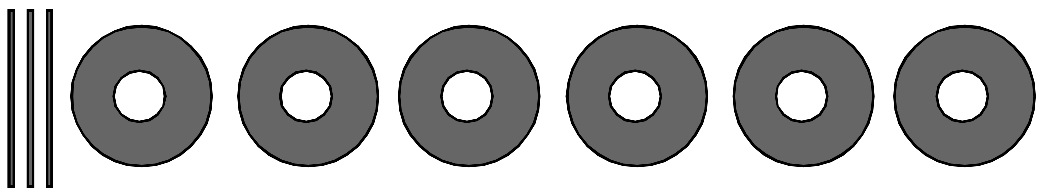
We are choosing four places of the 14 available to place the dividers.

Sticks and Stones

Counting

Example:
You are ordering a half-dozen doughnuts (6), and need to choose from among four flavors: Glazed, powdered, cream-filled, and jelly-filled. How many different doughnut orders are possible?

This would make for a very long list if we were to try and figure out how many ways she can order every combination of doughnut, so we apply the divider technique again. How many dividers do we need to separate the four flavors?



How many items are we arranging? Write the math.

If you got ${}_9C_3 = 84$, perfect!

Practice:

1. How many ways can 10 chocolates be divided among 8 students?

2. There are five items on the dollar menu at Burgerboy. You have enough to buy exactly five items. How many combinations of five items can you buy?

3. You are playing a racing video game. To begin, you get to adjust the tuning of your car by adding ten points to three categories. You can adjust your speed, handling, and acceleration by adding anywhere from 0 to 10 points to each category. How many tuning options are there for the car's initial setup?

4. How many ways can fifteen teddy bears be divided among six toddler beds at preschool if each must get at least one bear?

5. Uncle Henry is feeling more generous, and has decided this time to distribute four $1 bills, three $5 bills, two $20 bills, and a $50 bill to his five nieces and nephews. How many ways can he distribute the money this time?