Fibonacci and Lucas Numbers

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1 Fibonacci Numbers

1.1 Sum of Terms

The sum of the first \( n \) Fibonacci numbers is given as follows:

\[
F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1
\]

which we can easily prove with induction.

We use this to prove the following theorem:

**Theorem:** Let \( n \) and \( k \) be two positive integers. Prove that between the consecutive powers \( n^k \) and \( n^{k+1} \) there are no more than \( n \) Fibonacci numbers.

**Proof:** We proceed by contradiction. Assume that between some \( n^k \) and \( n^{k+1} \), there exist at least \( n + 1 \) Fibonacci numbers. Then we have,

\[
n^k < F_{r+1}, F_{r+2}, F_{r+3}, \ldots, F_{r+n+1}, \ldots < n^{k+1}
\]

The sum of the first \( n - 1 \) of these numbers is

\[
F_{r+1} + F_{r+2} + \cdots + F_{r+n-1} = F_{r+n-1} + F_{r+n-2} + \cdots + F_1 - (F_r + F_{r-1} + \cdots + F_1)
\]
\[
= F_{r+n+1} - 1 - (F_{r+2} - 1)
\]
\[
= F_{r+n-1} - F_{r+2}.
\]

Solving for \( F_{r+n+1} \) yields

\[
F_{r+n+1} = (F_{r+1} + F_{r+2} + F_{r+3} + \cdots + F_{r+n-1}) + F_{r+2}
\]

which is a sum of \( n \) Fibonacci numbers, each of which is greater than \( n^k \). Thus, \( F_{r+n+1} > n(n^k) = n^{k+1} \), contradicting our assumption.

1.2 Sum of Squares

The formula for the sum of square of the first \( n \) Fibonacci numbers is given as follows:

\[
F_1^2 + F_2^2 + F_3^2 + \cdots + F_n^2 = F_n F_{n+1}
\]

We can use induction to prove this, although there is a nice geometric proof for it. You should be able to figure it out with the diagram below:
1.3 Binet’s Formula

Binet’s Formula states that \( F_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right] \). We can elegantly derive this using the following lemma:

**Lemma:** If \( x^2 = x + 1 \), then, for \( n = 2, 3, 4, \ldots \), we have:

\[
    x^n = F_n x + F_{n-1}
\]

**Proof:** This is trivial for \( n = 2 \). Suppose that \( x^n = F_n x + F_{n-1} \) for some \( n > 2 \). Then:

\[
    x^{n+1} = x^n \cdot x = (F_n x + F_{n-1})x \\
    = F_n x^2 + F_{n-1}x \\
    = F_n (x + 1) + F_{n-1}x \\
    = (F_n + F_{n-1})x + F_n \\
    = F_{n+1} x + F_n,
\]

as desired.

The two numbers \( x \) which satisfy \( x^2 = x + 1 \) are \( \alpha = \frac{1 + \sqrt{5}}{2} \) and \( \beta = \frac{1 - \sqrt{5}}{2} \). Thus, for all \( n = 2, 3, 4, \ldots \), we have

\[
    \alpha^n = F_n \alpha + F_{n-1}
\]

and

\[
    \beta^n = F_n \beta + F_{n-1}.
\]

Subtracting these yields \( \alpha^n - \beta^n = F_n (\alpha - \beta) \). Noting that \( \alpha - \beta = \sqrt{5} \) yields Binet’s Formula.
2 Lucas Numbers

2.1 Introduction

Just as \( F_n \) denotes the \( n \)th Fibonacci number, we define \( L_n \) as the \( n \)th Lucas number. The Lucas sequence is defined by

\[
L_n = F_{n-1} + F_{n+1}.
\]

The first few Lucas numbers are 1, 3, 4, 7, 11, 18, 29, 47, 76, . . .

Since the Fibonacci numbers are generated by the recursion

\[
F_n = F_{n-1} + F_{n-2}
\]

the Lucas numbers also have that property, that is, for \( n > 2 \),

\[
L_n = L_{n-1} + L_{n-2}
\]

We define \( L_0 = 2 \) because \( L_2 = L_1 + L_0 \).

Note that a Lucas number is always greater than its corresponding Fibonacci numbers, except for \( L_1 \).

2.2 A Formula

We use Binet’s Formula to derive a formula for \( L_n \). We have:

\[
L_n = \frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} + \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}
\]

\[
= \frac{1}{\alpha - \beta} \left[ \alpha^n \left( \frac{1}{\alpha} + \alpha \right) - \beta^n \left( \frac{1}{\beta} + \beta \right) \right].
\]

Substituting \( \alpha = \frac{1 + \sqrt{5}}{2} \) yields \( \frac{1}{\alpha} + \alpha = \sqrt{5} = \alpha - \beta \), and similarly, \( \frac{1}{\beta} + \beta = -\alpha + \beta \), so the formula for the Lucas numbers is

\[
L_n = \alpha^n + \beta^n.
\]

2.3 Some Properties of Fibonacci and Lucas Numbers:

We can use Binet’s Formula to help us prove the following theorem:

**Theorem:** Let \((1 + 2x)^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \). Substitute \( F_k \) for \( x^k \) in this expression, yielding \( a_0 F_0 + a_1 F_1 + a_2 F_2 + \cdots + a_n F_n \). This sum is equal to \( F_{3n} \).

**Proof:** By the Binomial Theorem,

\[
(1 + 2x)^n = \sum_{k=0}^{n} \binom{n}{k} 2^k x^k.
\]

Denote \( S = a_0 F_0 + a_1 F_1 + a_2 F_2 + \cdots + a_n F_n \). Then,
\[ S = \sum_{k=0}^{n} \binom{n}{k} 2^k F_k \]
\[ = \sum_{k=0}^{n} \binom{n}{k} 2^k \left( \frac{\alpha^k - \beta^k}{\alpha - \beta} \right) \]
\[ = \frac{1}{\alpha - \beta} \left[ \sum_{k=0}^{n} \binom{n}{k} 2^k \alpha^k - \sum_{k=0}^{n} \binom{n}{k} 2^k \beta^k \right] \]
\[ = \frac{1}{\alpha - \beta} [(1 + 2\alpha)^n - (1 + 2\beta)^n] \]

Since \( \alpha^2 = \alpha + 1 \), we have
\[ 1 + 2\alpha = \alpha^2 + \alpha \]
\[ = \alpha(1 + \alpha) \]
\[ = \alpha^3 \]

Similarly, \( 1 + 2\beta = \beta^3 \). Thus,
\[ S = \frac{1}{\alpha - \beta} [(\alpha^3)^n - (\beta^3)^n] = \frac{\alpha^{3n} - \beta^{3n}}{\alpha - \beta}, \]
which is just \( F_{3n} \) by Binet’s Formula.

We can continue with this. Because \( L_n = \alpha^n + \beta^n \), we have:
\[ \sum_{k=0}^{n} \binom{n}{k} 2^k L_k = L_{3n} \]
and
\[ \sum_{k=0}^{n} \binom{n}{k} L_k = L_{2n} \]

Additionally, we have \( F_{2n} = F_n L_n \), which we can prove using the formulas for \( F_k \) and \( L_k \).

Another theorem regarding the Lucas and Fibonacci numbers is:

**Theorem:**

\[ F_{m+p} + (-1)^{p+1} F_{m-p} = F_p L_m. \]

This is easily proven.

**Proof:** We wish to show that
\[ \frac{\alpha^{m+p} - \beta^{m+p}}{\alpha - \beta} + (-1)^{p+1} \frac{\alpha^{m-p} - \beta^{m-p}}{\alpha - \beta} = \frac{\alpha^p - \beta^p}{\alpha - \beta} (\alpha^m + \beta^m) \]

Then
\[ \alpha^{m+p} - \beta^{m+p} + (-1)^{p+1}(\alpha^{m-p} - \beta^{m-p}) = \alpha^{m+p} + \alpha^p \beta^m - \alpha^m \beta^p - \beta^{m+p} \]

\[ (-1)^{p+1}(\alpha^{m-p} - \beta^{m-p}) = \alpha^p \beta^m - \alpha^m \beta^p \]

\[ (-1)(\alpha \beta)^p (\alpha^{m-p} - \beta^{m-p}) = \alpha^p \beta^m - \alpha^m \beta^p \]

\[ -(\alpha^m \beta^p - \alpha^p \beta^m) = \alpha^p \beta^m - \alpha^m \beta^p \]

\[ \alpha^p \beta^m - \alpha^m \beta^p = \alpha^p \beta^m - \alpha^m \beta^p, \]

as desired.

We use this to prove yet another property:

**Theorem:** The sum of any 4n consecutive Fibonacci numbers is divisible by \( F_{2n} \).

**Proof:** Let \( S \) equal this sum. Then we have:

\[ S = \sum_{k=a+1}^{a+4n} F_k \]

\[ = s_{a+4n} - s_a \]

\[ = (F_{a+4n+2} - 1) - (F_{a+2} - 1) \]

\[ = F_{a+4n+2} - F_{a+2} \]

Letting \( m = a + 2n + 2 \) and \( p = 2n \) yields

\[ S = F_{a+4n+2} - F_{a+2} = F_{2n} L_{a+2n+2}, \]

completing our proof.

One more theorem regarding the Lucas numbers:

**Theorem:** Suppose \( p > 3 \) is a prime number, and \( p^k \) is a positive integral power of it. Then the \( 2p^k \)th Lucas number \( L_{2p^k} \) ends in a 3.

**Proof:** Numbers of the form \( 6m, 6m + 2, 6m + 3, 6m + 4 \) are always composite for \( m > 0 \), so a prime \( p > 3 \) must satisfy

\[ p \equiv \pm1 \pmod{6} \]

so

\[ p^k \equiv \pm1 \pmod{6} \]

which implies \( p^k = 6n \pm 1 \) for some integer \( m \). Thus, \( 2p^k = 12m \pm 2 \).

The Lucas numbers end in the digits

\[ 1, 3, 4, 7, 1, 8, 9, 7, 6, 3, 9, 2, 1, 3, 4, 7, \ldots \]

which repeats with period 12:

\[ 1, 3, 4, 7, 1, 8, 9, 7, 6, 3, 9, 2 \]
The second and tenth numbers are 3’s, so counting along to the $12m \pm 2$th Lucas number will always yield a units digit of 3.