MATHCOUNTS

2008
■ Yongyi's National Competition #2 ■
Sprint Round
Problems 1–30

Name ____

State ____

DO NOT BEGIN UNTIL YOU ARE INSTRUCTED TO DO SO.

This round of the competition consists of 30 problems. You will have 40 minutes to complete the problems. You are not allowed to use calculators, books, or any other aids during this round. If you are wearing a calculator wrist watch, please give it to your proctor now. Calculations may be done on scratch paper. All answers must be complete, legible, and simplified to lowest terms. Record only final answers in the blanks in the right-hand column of the competition booklet. If you complete the problems before time is called, use the remaining time to check your answers.

In each written round of the competition, the required unit for the answer is included in the answer blank. The plural form of the unit is always used, even if the answer appears to require the singular form of the unit. The unit provided in the answer blank is the only form of the answer that will be accepted.

Total Correct	Scorer's Initials
30	**YC**

1. If $\sqrt{20} + \sqrt{45} = \sqrt{x}$, what is the value of x?

1. 125

2. Connie has 5 times the amount of money as Bill has, and Steven has 35 times the amount of money as Bill has. If Connie has $191, how much money, in dollars, does Steven have?

2. $ 1337

3. Three dice, each with an equal number of faces numbered consecutively starting with 1, are rolled. The probability that at least one die shows a 1 on the top is 169/512. How many faces does each die have?

3. 8 faces

4. A cube is sliced into eight smaller cubes by three perpendicular cuts. If the combined surface area of the eight small cubes is 768 square inches, what is the surface area, in square inches, of the large cube when the pieces are reassembled together?

4. 384 square inches

5. The area of an equilateral triangle is numerically equal to its side length. What is this number? Express your answer as a common fraction in simplest radical form.

5. $\frac{4\sqrt{3}}{3}$

6. How many positive integers between 0 and 100, inclusive, have no prime factor greater than 5?

6. 34 positive integers

7. A positive integer n leaves a remainder of 2 when divided by 5 and a remainder of 7 when divided by 9. If n is less than 100, what is the maximum possible value of n?

7. 97

8. The area of a square is exactly twice the area of a circle. What is the ratio of the side length of the square to the diameter of the circle? Express your answer as a common fraction in simplest radical form and in terms of π.

8. $\frac{\sqrt{2\pi}}{2}$

9. In the equation $y = \frac{2x+1}{3x-2}$, a value of x and a value of y are not attainable. What is the value of $x - y$?

9. 0

10. The word YONGYI has two Y's. How many ways are there to arrange the letters if the two Y's cannot be next to each other?

10. 240 ways

11. The 2008th row of the Pascal Triangle begins with 1, 2007, … What is the remainder when the sum of all the terms in that row is divided by 1000?

11. 128

12. The value of 2008^5 is written as a product of 20 prime numbers, not necessarily distinct. What is the sum of the 20 numbers used in this product?

12. 1285

13. The ratio of the surface area of a sphere to its volume is 1/2. What is the radius of the sphere?

13. 6

14. A semicircle of radius 10 has center C and endpoints A and B. Segments CA and CB are joined such that the resulting figure is a cone. What is the volume of this cone? Express your answer as a common fraction in simplest radical form and in terms of π.

14. $\frac{125\pi\sqrt{3}}{3}$

15. Two dice, each with three faces painted green, two faces painted blue, and one face painted red, are rolled. What is the probability that the colors on the top of the dice are different?

15. $\frac{11}{18}$

16. A spherical meteor that hit Mars left a crater 40 miles wide and 8 miles deep. What is the radius, in miles, of the meteor?

16. 29 miles

17. A triangle is formed from three points that are randomly chosen on the graph of $y = \sqrt{16 - x^2}$. What is the minimum possible measure of the largest angle of the triangle, in degrees?

17. 90 degrees

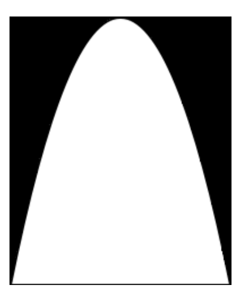
18. A function $f(x)$ is defined over the positive real numbers such that $f(2x) = 2f(x) + 1$. If $f(8) = 16$, what is the value of $f(1)$? Express your answer as a common fraction.

18. $\frac{9}{8}$

19. A cup is in the shape of an inverted truncated cone. The bottom of the cup is a circle with radius 2 inches and its top is a circle with radius 5 inches. If the cup is 9 inches high, what is the volume, in cubic inches, of the cup? Express your answer in terms of π.

19. 117π cubic inches

20. A train is traveling through a tunnel with a parabolic archway that is 20 feet wide and 25 feet tall. If the train is 8 feet wide, what is the maximum possible height, in feet, of the train so that it can still pass through the tunnel?



20. 21 feet

21. A *nested radical* is a radical that appears in another radical, as in $\sqrt{3+2\sqrt{2}}$. Often, they can be simplified. For example, $\sqrt{3+2\sqrt{2}} = 1+\sqrt{2}$. What is the simplified value of $\sqrt{31+12\sqrt{3}}$? Express your answer in simplest radical form.

21. $2+3\sqrt{3}$

22. The *digital root* of a positive integer n is calculated by repeatedly taking the sum of digits of n until it becomes a one-digit number. For example, the digital root of 88888888 is 1 because $8+8+8+8+8+8+8+8=64$, $6+4=10$, and $1+0=1$. What is the digital root of 2008^{2008}?

22. 1

23. A positive integer n greater than 2 has the property that the interior angle measure of a regular polygon with n sides is divisible by n. What is the sum of all possible values of n?

23. 9

24. Let a, b, c be the three solutions to the equation $x^3+4x^2+x-6=0$. What is the value of $(a+1)(b+1)(c+1)$?

24. 4

25. A *semiprime* is a positive integer that is the product of two prime numbers, not necessarily distinct. For example, 57 is a semiprime because it is the product of 3 and 19, both of which are prime numbers. In addition, all numbers that are squares of primes are semiprimes by definition. How many positive integers below 1000 are semiprimes?

25. 299

26. Express $4.134134134\ldots_5$ as a common fraction in base 10.

26. $\frac{135}{31}$

27. Two gears are specially constructed such that one has exactly three times as much teeth as the other. If the larger gear is rotated around the smaller gear, which remains stationary, how many times would the larger gear have rotated around its own axis? Express your answer as a common fraction.

27. $\frac{4}{3}$ times

28. In a regular hexagon, all the diagonals connecting two nonadjacent vertices are drawn. The diagonals divide this hexagon into r regions. If n diagonals were drawn, what is the value of $n + r$?

28. 33

29. There are many values a and b for which $a^2 = 2b^2 + 1$. For example, $3^2 = 2 \cdot 2^2 + 1$ and $17^2 = 2 \cdot 12^2 + 1$, which are represented by the ordered pairs (3, 2) and (17, 12). What is the next smallest pair of values (a, b) that satisfies the above equation? Express your answer as an ordered pair.

29. (99 , 70)

30. Tom has a strange calculator that only has two buttons: [+1] and [×2]. The [+1] button adds the current number on display by 1 and the [×2] button multiplies the current number on display by 2. If the calculator starts on the number 1, let the maximum number of buttons he needs to push to reach 1337 be a and the minimum number of buttons he needs to push to reach 1337 be b. What is the value of $a - b$?

30. 1321

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2008
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Target Round
Problems 1 and 2

Name ____

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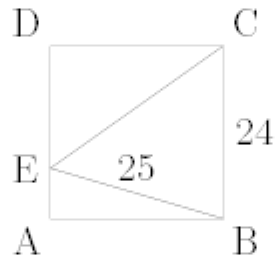
This section of the competition consists of eight problems, which will be presented in pairs. Work on one pair of problems will be completed and answers will be collected before the next pair is distributed. The time limit for each pair of problems is six minutes. The first pair of problems is on the other side of this sheet. When told to do so, turn the page over and begin working. Record only final answers in the designated blanks on the problem sheet. All answers must be complete, legible, and simplified to lowest terms. This round assumes the use of calculators, and calculations may also be done on scratch paper, but no other aids are allowed. If you complete the problems before time is called, use the time remaining to check your answers.

Total Correct	Scorer's Initials
2	**YC**

1. Square ABCD has side length 24 and BE = 25. What is the length of CE? Express your answer as a decimal to the nearest hundredth.



1. 29.41

2. If there are exactly two solutions to the equation $x^4 - 6x^3 + nx^2 - 6x + 1 = 0$, what is the value of n?

2. 11

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Problems 3 and 4

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2	**YC**

3. How many positive integers n less than 10000 have the property that $2n$ is a perfect square, $3n$ is a perfect cube, and $4n$ is a perfect fourth power?

3. 0 positive integers

4. What is the positive difference between the area of the largest equilateral triangle and the area of the smallest equilateral triangle that can be inscribed in a square of side length 4? (Inscribed means that all three vertices of the equilateral triangle are on the boundary of the square.) Express your answer in simplest radical form.

4. $28\sqrt{3} - 48$

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Problems 5 and 6

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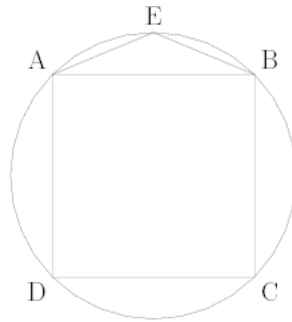
5. Square ABCD is inscribed in a circle. Isosceles triangle ΔAEB is then constructed with E on minor arc $\widehat{AB}$. What is the number of degrees in the measure of ∠ABE? Express your answer as a decimal to the nearest tenth.



5. 22.5 degrees

6. A right circular cone with a height of 100 feet has a volume of 30,000 cubic feet. This cone is partially submersed underwater with its base parallel to the surface of the water. If the portion of the cone that is above water has a volume of 10000 cubic feet, what is the height, in feet, from the base of the cone to the surface of the water? Express your answer as a decimal to the nearest hundredth.

6. 30.66 feet

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Problems 7 and 8

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7. For positive integers n, denote $\mathrm{S}(n)$ to be the sum of digits of n. What is the smallest possible value of n for which $\mathrm{S}\big(\mathrm{S}(n)\big) \geq 10$?

7. 199

8. An army of cadets always marches at a constant speed in a square formation, with each side measuring 50 feet. Its mascot, a dog, starts at the lower-right corner and trots counterclockwise around the perimeter of the square while the army is marching. By the time the dog reaches its starting point, the army has advanced 50 feet. What is the total distance, in feet, the dog traveled? Express your answer as a decimal to the nearest hundredth.

8. 209.06 feet